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Mathematical Excursions to the World's Great Buildings

by Alexander Hahn

PRINCETON AND OXFORD: PRINCETON UNIVERSITY PRESS, 2012, 318 PP., US $49.50, ISBN 978-0-691-14520-4

REVIEWED BY STEPHEN R. WASSELL AND KIM WILLIAMS

SRW and KW: This is a book by a mathematics professor at the University of Notre Dame, using his formidable expertise to explore and clarify aspects of some of the great icons of Western architecture. Notre Dame has a fine architecture school, among the best in the United States for those interested in classical architecture, and the book is conceived as the textbook for a class that Professor Hahn has been teaching for several years to undergraduate architecture students as a second mathematics course after Calculus I. We decided to approach the review as a debate, with one of us (SRW) on the side of the mathematics professor and the other (KW) on the side of the architecture student. Questions that naturally arise include: (1) Is this the mathematics that architecture students need to know? (2) Will architecture students believe this? (3) Could this book be used for other types of courses, for example, for a "quantitative literacy" course within a general education program? We will address these questions at the end of the review.

The structure of the book is historical and chronological. This is certainly fitting for the discussions of architecture, because generally speaking the architecture of any given period is in some way a reflection of what has preceded it—if not directly an outgrowth, then perhaps a rediscovery or even a reaction against it. But Hahn himself points out a noticeable disjuncture between the architectural narrative, which flows rather straightforwardly, and the mathematical narrative that accompanies it: "the chronological alignment between mutually informing architecture and mathematics is rare" (p. vii).

KW: First let me say that this is certainly a great step forward from what I was offered in the way of mathematics 40 years ago, when the architecture program I was enrolled in required one precalculus course and one calculus course, with little or no attempt that I can recall to link that material to the interests of the architecture student. Although I began architecture school the year just after the inauguration of the Sydney Opera House, a building examined at length in this book, there were no particular incentives that encouraged us to go further into matters of mathematics and architecture. Such matters were considered to be the realm of the engineer.

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The architect's role was to conceive the grand design; the engineer's role was to make it stand up. All of us wanted to become Jørn Utzon (the designer of the Sydney Opera House), but I don't remember any of us wanting to become Ove Arup (the engineer who played a crucial role). Still, as the years have gone by and my perspectives have altered (matured, I like to think), I have come not only to see the value of mathematics, but also to feel a fascination for it in its own right. And there is a definite need for textbooks on architecture and mathematics. But I have some objections to how the architecture and mathematics are intertwined here. I'll make these clear as we go forward.

SRW: This is a highly interdisciplinary book, touching on such topics as architectural history, math history, mathematics, and even some aspects of the general history of society and culture. (The focus is on Western cultures, with the usual inclusion of key influences from the Middle East, on both architecture and mathematics.) I also liked seeing the etymology of selected words sprinkled throughout the text. But unless the book were to be several times longer than it is, there is no way that it would be able to cover any one of these disciplines in great detail; this may be seen as a drawback by some readers.

SRW and KW: The first chapter eases one into the book in a gentle way, making connections between the prehistory of mathematics and the prehistory of architecture as similar reflections of man's growing awareness of himself and his surroundings and the founding of civilizations. The math problems that accompany this chapter aren't likely to pose much problem for first- or second-year college students of any major.

KW: The second chapter, "Greek Geometry and Roman Engineering," gives me pause. Presented sequentially as they are, there seems to be an underlying assumption that Roman architecture, with its classical motifs and use of the orders, was a natural development of the Greek tradition, but in fact the two architectures were quite different. Greek architecture is a post-and-beam system, whereas Roman architecture is a wall architecture. The Romans made extensive use of arches and domes, where the Greeks did not, but this is not because the arch and vault were Roman inventions (even the Egyptians knew about the arch, and the Greeks knew it, but preferred not to use it). The chapter begins with an introduction to the theory of forces, the effects of which "the architects of antiquity were aware … and had an intuitive understanding of …," but "the conceptual challenges that the analysis of forces involves were beyond the reach of Greek builders and Roman engineers" (p. 25). Putting the lumping together of two periods aside, neither age appears to have been hampered in achieving outstanding architecture by the lack of the analysis of forces. In fact, adequate knowledge of the tensile and compressive forces was not acquired until the age of Galileo, yet countless structurally sound arches and domes were built without that knowledge. Romans certainly appear to have had knowledge of the outward thrust of arches and domes; they solved the problem by throwing mass at it, hence the enormously thick walls of the Pantheon, for instance. That massiveness became in turn one of the aesthetic hallmarks of Roman architecture, and certainly one suited to demonstrating the strength of an empire. Are we to conclude that if they had been able to explain the thrust in quantitative terms that their architecture would have been *better*?

SRW: Although I understand KW's points, I did not have the same reaction when I read the second chapter. I certainly did not perceive any implication on Hahn's part that knowledge of the mathematics of forces would have aided Roman architects to design any better than they did.

My main reaction to the second chapter is that this is where the reader first realizes that the text is foremost a mathematics text. Some basics from Euclid's *Elements* are reviewed, such as the construction of an equilateral triangle, the bisection of an angle, and the cutting of a line segment in extreme and mean ratio (which defines what we now call the golden ratio or golden section). Regular polygons are defined, including discussions of central angles and vertex angles, and the concept of straightedge and compass constructions is introduced. Right-triangle trigonometry is also developed. And of course, the analysis of forces, discussed earlier by KW, involves a good deal of mathematics.

My bachelor's degree was in architecture, and we were required to take a separate course on structural analysis. I don't know what the norm is now, but if structures aren't covered anywhere else in an architecture school's curriculum, then certainly the exposure to force diagrams and analyses that a student would obtain from this text, starting in Chapter 2, would be quite valuable.

KW and SRW: Chapter 3, appropriately entitled "Architecture Inspired by Faith," is a compilation of Byzantine, Islamic, Romanesque, and Gothic architecture. This combination is more than just a convenience in the organization of the book; the truth of the matter is that the labels and divisions that later cultures place on earlier ones are not always as straightforward as they may be presented. Key architectural concepts are discussed, such as the pendentive and the pointed arch, including a simplified force analysis of the latter. One highlight of this chapter is a discussion of the Milan Cathedral's geometry based on the annals of a building council that was set up to oversee its design and construction. The exercises at the conclusion of Chapter 3 are a continuation of the analysis of forces, with a discussion of symmetry followed by brief synopses on Norman architecture and medieval building practices. Although the discussion on symmetry essentially includes an introduction to group theory, the math-to-architecture ratio in Chapter 3 is smaller than it is in Chapter 2.

SRW: Chapter 4, "Transmission of Mathematics and Transition in Architecture," brings mathematics—and math history—back to the forefront. Hahn provides a concise history of number systems, including the Pythagoreans' knowledge of the incommensurability of the diagonal of a square, the drawbacks of Roman numerals, the role of Fibonacci in transmitting the Arabic numerals to the West, and the work of Stevin in the adoption of decimal fractions. Coordinate systems in two and three dimensions are introduced, as well as the distance formula and equations for lines, parabolas, ellipses, circles, and spheres. All of this leads up to a structural analysis of the Duomo of Florence. Embedded in the exercises at the end of the chapter is a discussion of the five Platonic solids.

KW: Chapter 4 presents what is to me a remarkable example of chronological nonalignment, that of the coordinate geometry of Descartes and Fermat of the seventeenth century, and the fifteenth-century dome of the Cathedral of Santa Maria del Fiore by Brunelleschi. "The purpose of this chapter is to tell the story of coordinate geometry … and, finally, to illustrate how this study clarifies the rising shape of the octagonal dome of the Cathedral of Florence" (p. 99). Thus we begin with Apollonius and Ptolemy, go on to the discovery of the coordinate plane by Descartes and Fermat, and finally to the discovery of coordinate space. This knowledge is then applied to visualization of the structure first of the hemispherical dome of the Hagia Sophia as a test case, and then to an understanding of the pointed octagonal dome of Santa Maria del Fiore. And this is where I found myself beginning to object. First of all, the process of construction described is ideal, not real. "The key to its geometry is the inner edge of the octagonal drum" (p. 120). But the actual geometry wasn't as pure and clean as it should have been; the octagonal drum was actually quite irregular, as Hahn notes later, with its longest and shortest sides differing by 2 feet. But that's not the only thing that doesn't add up: no one on a building site would work out such a complicated formula as that given on p. 121 to arrive at an approximate length of rope needed for the construction.

But my main objection is the use of a tool developed two centuries after the architecture it is used to illustrate. Now allow me to go back to the sentence that has been haunting me since the book's beginning: "The reality is that the elementary mathematics that might have clarified the understanding of a complex structure was almost always beyond the reach of the builders of the time" (p. vii). Coordinate geometry may help *us* to understand the dome as a complex object after the fact, but it obviously played no role in its conception or execution. Brunelleschi didn't require its help to understand the structure. Further, coordinate geometry wasn't just beyond the reach of the builders of the time; it was beyond the reach of everyone of the time. To me the disjuncture is just too great. If I were an architecture student, I don't think I would be convinced that this was something I needed to know.

However, if Hahn had moved his discussion of coordinate geometry forward to the 1600s when Descartes and Fermat were working on it, he could have delved into the techniques of stereotomy then being used by architects such as Guarino Guarini and Christopher Wren to cut stone to precise shapes for assembly on the worksite in projects for magnificent domes such as those of San Lorenzo and the Chapel of the Shroud (ca. 1668) in Torino and St. Paul's in London (1677). The architectural foment of that moment gave rise to mathematical research, as in the case, for instance, of Desargues with his sketch for *Brouillon Projet* (1640). Unfortunately, what architects were doing with stone at that time was beyond the reach of the mathematicians of the day! Interest in stereotomy would lead to research on curves with double curvature (for instance, Clairaut, *La recherche sur les courbes à double courbure*, 1731), which had profound influence on stonecutting treatises (Frezier, *La théorie et la practique de la coupe des pierres et des bois, ou traité de stéréotomie a l'usage de l'architecture*, 1737-1739), allowing architects and builders to define ever more complex shapes that could be built. This finally led to a new branch of geometry, with Monge's *Géométrie Descriptive* of 1798, a mathematics textbook developed for training architects and engineers. If I were an architecture student, this would be mathematics I'd think I ought to know, especially because the lessons learned from descriptive geometry lend themselves well to further developments in CAAD, which is an essential part of the architect's toolkit today.

KW: Chapter 5, "The Renaissance," covers the span from the early fifteenth century of Brunelleschi and Alberti through Palladio, Leonardo, Bramante, Michelangelo to Bernini in the seventeenth. A theme throughout this chapter is the use of circles and circular geometry. The symbolism of the circle as a "perfect" geometrical figure is tied into the use of centrally planned church architecture as a representation of the cosmos and its role as a "guide for important structural aspects" (p. 156). There is a lengthy discussion about the history of the design of St. Peter's in Rome, with the accent placed on "mathematical" features such as symmetry, symbolism, and shape. The discussion of circular geometry gives way to elliptical geometry as completion of St. Peter's Square moved into the 1600s. This chapter includes a long excursus on the mathematics of perspective, which can be skipped, we are told, because it is technical and does not impact the rest of the book. That may be, but the fact is that the discovery and development of linear perspective is one of the rare cases of genuinely parallel developments in architecture and mathematics, so why hasn't the author instead focused on that? The other discussions include the orders (construction of volutes) and the problem of raising an obelisk.

SRW: The mathematical analysis of perspective is indeed technical, but for me it was quite compelling. The basic goal is to analyze the ellipses that arise from circles seen in perspective. Hahn starts with the 6×6 square grid made canonical by Alberti in his treatise *On Painting*, the first treatise to include a description of the construction of a perspective drawing. Hahn places into Euclidean 3-space the floor plan of the 6×6 square grid, positioned on the xy-plane, and the canvas (i.e., the picture plane), positioned on the xz-plane. He then works out the mathematics, using parametric equations in 3-space, allowing one to relate points in the floor plan to their corresponding points on the canvas. One benefit is seeing how the math confirms the elegant simplicity of the formulas for various key points on the canvas, such as the vanishing point and the diagonal vanishing point. Moreover, Hahn uses the mathematical machinery he has developed to verify that the equation of a circle in the floor plan maps to the equation of an ellipse on the canvas. The mathematics is tedious and even a bit "cumbersome" (p. 185), as Hahn admits, but perhaps it will inspire the students who study from his text to persevere!

At the end of the chapter are several related exercises that comprise a series of rich studies. A few use a precisely drawn figure of the floor plan and the corresponding canvas for a one-point perspective of the canonical 6×6 square grid. This is a sort of template for the student to photocopy and then use to draw, in accurate perspective, lines at various angles to the grid, as well as a regular octagon. Another pair of exercises analyzes the grid of ellipses on the canvas corresponding to a grid of circles in the floor plan. Hahn makes the wise choice to draw 9 circles on the 6×6 square grid rather than 36, one circle for every four squares, with the center of each circle lying on the central point of its four-square pattern, which leads to whole-number radii in the mathematical analysis. One benefit of this analysis is the realization of a fact that was new to me: the circles in the grid seen at an angle (which is true except for the circles dead ahead of the artist's eye) correspond to ellipses on the canvas that have *nonhorizontal* focal axes. This level of mathematics can inspire a math major (or professor!), but I can certainly see why this section is labeled as optional.

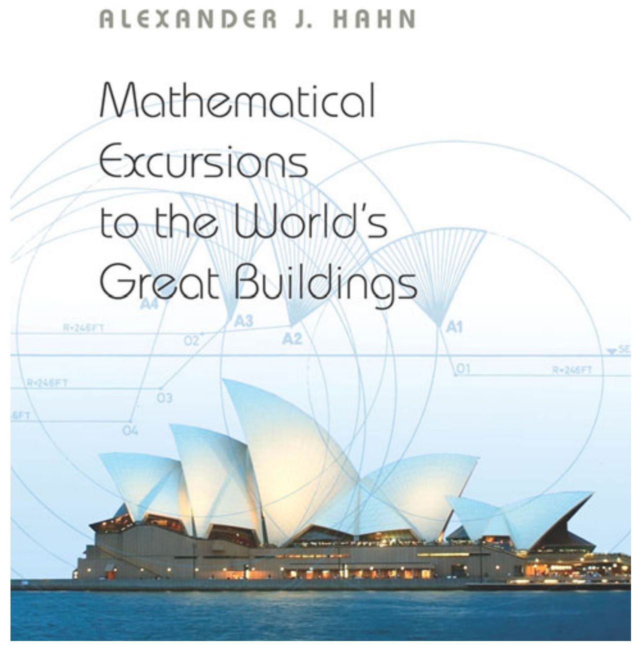
As a quick note, I found the discussion of music theory, with the inclusion of harmonics (often left out in such discussions), especially nice.

KW: Chapter 6, "A New Architecture," takes up with the Industrial Revolution. A section dealing with Coulomb's theories of statics is followed by a long investigation of the Sydney Opera House. By this point, however, the book seems to me to have given up on being about the use of mathematics to "read" architecture and has turned into a standard book on structures. What is the difference? The difference is that the first kind of book examines the shapes and arrangements and agglomerations of shapes that make up architecture and helps us both to see how mathematics can be used in architecture as a form generator, and to infuse buildings with the knowledge possessed by a given civilization. In the second kind, math is used as a quantifier to determine weights, volumes, and behaviors of building materials.

SRW: I certainly agree with KW that Chapter 6 is heavy on structural analysis, to the point that it takes on a different flavor than the rest of the book. Of course, it may be very valuable for a student of architecture to see somewhat advanced force diagrams, especially if this is the only course in the curriculum to include them.

The discussion of the evolution of dome structures at the start of the chapter is very interesting (St. Paul's in London, the Panthéon in Paris, St. Isaac's in Saint Petersburg, and the Capitol Building in Washington, DC, with comparisons to St. Peter's in Rome, which had already been analyzed in Chapter 5). And mathematicians will certainly appreciate the evolution of the Sydney Opera House's design, the upshot being that all of the now famous sail-like surfaces consist of spherical triangles *drawn from the same sphere*, which made it possible to standardize the construction of the structure enough to solve the problem of skyrocketing costs that had already been getting out of hand.

SRW: Whereas Chapter 6 has a different flavor from the rest of the book, Chapter 7 is on another level entirely. It starts with a review of differential and integral calculus, which takes about a dozen pages, half of the chapter. The applications of calculus in the second half of the chapter include finding the volumes of some spherical domes (the Hagia Sophia in Istanbul and the Pantheon in Rome), determining that the shape of an ideal arch is a catenary curve, and analyzing moments of force and centers of mass. Although the review of calculus is nicely done, the reader certainly needs to have seen the subject before for it to be understandable (as Hahn notes), and the applications of calculus require some math savvy. We are warned in the Preface that Chapter 7 should be



viewed as optional (along with some of the more difficult sections in Chapters 5 and 6), and, depending on the mathematical maturity of the students, the professor may very well want to skip them.

KW: In Chapter 7 Hahn enters a subject he knows thoroughly and loves: calculus and its applications. Again we are told this can be skipped. I can't help thinking that the mathematician here is in some way talking down to the architects. Although it is true—as even SRW admits—that the mathematics presented in the more technical sections, and especially in Chapter 7, is challenging and requires savvy, I feel that, because the book is a course for architecture students, these challenges should be presented as ones that are approachable and appropriate for the architect. All too often in mathematics courses for architects, the architects attempt to keep the mathematics but throw away the mathematicians. Here we have a splendid opportunity for the mathematician to present not only his view of architectural problems, but also his enthusiasm for how they can be approached. Yet when the going looks tough, the architect, rather than being encouraged to face the challenges, is given an out. Today, however, advanced techniques of mathematics and computer science are being increasingly applied to generate architectural form and aesthetics as well as the structural features to make that form buildable. Architects shouldn't be encouraged to back away from that. Otherwise the architecture profession risks being relegated while engineers, presumed capable of understanding the technical aspects, assume increasingly important roles on the building team.

KW and SRW: The book is nicely and copiously illustrated, the research well attended to. The author has been careful not to put words into the mouths of architects or to make untoward speculation about "how they did it." The mathematics exposition is clearly written and precise throughout, and the quantity and quality of the exercises are highly commendable. As for the questions we raised at the beginning:

1. Is this the mathematics that architecture students need to know?

KW: Yes, at least some of it.

SRW: Although I do not think that this was conceived as a "mathematics for architects" sort of text, it does contain many good examples of the use of math not only to analyze structures but also to communicate about the aesthetics of design, which I think is just as important. It touches on such topics as symmetry groups, descriptive geometry, and vector decomposition, making it a good complement to Calculus I, a common math requirement for architecture students.

2. Will architecture students believe this?

KW: Well, if I were an architecture student, I think I'd come out of the course sure that architecture is a fascinating subject, and sure glad I didn't have to be distracted from it any longer by that math stuff. But this is partly due to the book's attitude, which I mentioned earlier, that assumes that some of the math presented will be too "technical" for the architect. If the book encouraged the architecture student to apply himself to understanding the mathematics presented, then the sense of accomplishment when that was achieved might be a reward in itself.

SRW: This would depend in large part on exactly what material is taught and how well it is presented, of course. The teacher needs to stress that the overriding goal is to exercise one's ability to think about objects in space and to explore their characteristics through the use of mathematics. In any case I definitely think that conscientious students would certainly value most of the material from the nonoptional parts of the text (and I'm confident, through communications with the author, that there is plenty of material in these parts to make a semester-long course).

3. Could this book be used for other types of courses, for example, a "quantitative literacy" course within a general education program?

SRW: Yes, and in my opinion math professors will be very comfortable with the material in this book. First, they will not feel overwhelmed by the amount of nonmathematical material they need to learn; a basic understanding of architecture and architectural history will suffice, and the textbook itself will fill in the rest. Second, they will feel that the level of math warrants a college-level quantitative literacy designation; many publications, both books and articles, on the relationships between architecture and math are a bit soft on the math, but this text contains plenty of it at the college level.

KW: The problem that will be faced in using this book for quantitative literacy courses for nonarchitecture majors will be that there is even less general literacy about architecture than there is about mathematics. Nonarchitecture majors will have had at least some basic high-school mathematics (geometry, algebra, and trigonometry), but they are almost sure to have no basic knowledge about architecture at all. That means that the book has to educate in not one but two disciplines simultaneously. The architecture glossary provided at the end of the book provides some support in this regard (in fact, I have to wonder why, if the book is intended for architecture students, there is not a glossary of mathematics terms, which are likely to be less familiar to them than the architecture terms).

KW and SRW: This well-written, well-intentioned book takes an important step forward in showing the relevance of mathematics and its history to architecture and its history. One of its strengths is that it ties together the histories of math, architecture, engineering, and culture in general, allowing for a more holistic understanding of some key periods. For example, the connections and transitions from the early Renaissance to the high Renaissance, then to the Baroque period, and into the Enlightenment, are made more understandable through Hahn's interweaving of the oft-separated histories of different fields. Hahn has a website (http://www3.nd.edu/~hahn/math10270/) to complement this text for the class he teaches, and he has made it open for anyone to see. The resources made available there will certainly aid any professor who wants to adopt this textbook for a class, whether for architecture majors or a general audience.

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