MATH 80220: TOPICS IN ALGEBRA
SOERGEL BIMODULES

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Finite Coxeter groups are the finite groups generated by reflections on finite-dimensional real vector spaces, where a reflection is a linear map of order two which fixes a hyperplane pointwise. They include the symmetry groups of regular polytopes (such as regular polygons, for which the symmetry groups are finite dihedral groups, and Platonic solids) and the Weyl groups of semisimple complex Lie algebras (such as $\mathfrak{sl}(n, \mathbb{C})$, which is the Lie algebra of the special linear group $\mathrm{SL}(n, \mathbb{C})$ and has the symmetric group S_n as its Weyl group). General (possibly infinite) Coxeter groups have realizations as discrete real reflection groups and may be described algebraically as certain natural "amalgamations" of (finite or infinite) dihedral groups.

The crystallographic Coxeter groups (roughly, those which may be regarded as reflection groups on free abelian groups) have significant applications to algebra, geometry, combinatorics and representation theory, generalizing applications of Weyl groups to the study of semisimple complex Lie algebras and algebraic groups. These applications have been greatly extended since the late 1970s following the discovery by Kazhdan and Lusztig of certain polynomials, defined using Hecke algebras, attached to pairs of elements of a Coxeter group. Answers to many questions of geometric or representation theoretic interest (e.g. local intersection cohomology of Schubert varieties and composition factor multiplicities in Verma modules) may be determined from these Kazhdan-Lusztig polynomials. The classical instance of this was the Kazhdan-Lusztig conjecture, which determined those composition factor multiplicities for semisimple complex Lie algebras. Its original proof used a variety of deep results and ingredients (e.g. D-modules, perverse sheaves, Riemann-Hilbert correspondence and machinery from the proof of the Weil conjectures) and showed incidentally that Kazhdan-Lusztig polynomials for finite Weyl groups have non-negative coefficients.

In the 1990's, Soergel developed a more elementary conjectural approach to proof of the Kazhdan-Lusztig conjecture based on the study of certain bimodules, now called Soergel bimodules, over the symmetric algebra of the real vector space on which a Coxeter group acts as reflection group. In the mid 2010's, Elias and Williamson completed this program by a study of Hodge theory of Soergel bimodules, thereby also completing the proof for general Coxeter groups that Kazhdan-Lusztig polynomials have non-negative coefficients and that structure constants of Hecke algebras enjoy many remarkable positivity properties.

This course will survey these and related developments, largely following the book "Introduction to Soergel bimodules" by Elias, Makisumi, Thiel and Williamson. It will not be possible to provide full details of the background, proofs or applications. However, we will aim to keep the course as elementary as possible (but not more so).

This is an advanced graduate course, for which Basic Algebra I and II (Math 60210-60220) are prerequisite and for which any additional familiarity with elementary category theory, geometry and topology, commutative algebra, algebraic geometry and homological algebra etc would be very desirable. Undergraduate students interested in taking the course should consult with the instructor to determine if it is sensible for them to do so, given their background.