Topics in Mathematical Physics, Fall 2022
MATH 80870
"Introduction to Conformal Field Theory"
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Two-dimensional conformal field theory (CFT) was developed in 1980's starting from the seminal work of Belavin-Polyakov-Zamolodchikov (1984). It studies quantum field theory as a set of correlation functions on punctured Riemann surfaces, with punctures decorated by elements of a vector space V (the "space of fields" or "space of states"), subject to constraints imposed by conformal symmetry (in particular, the moduli space of complex structures on the surface plays a central role). Symmetry induces a rich structure on V and on correlators. In particular, V has to carry a representation of a special infinite-dimensional Lie algebra – the Virasoro algebra. Conformal field theory is a setup for quantum field theory in which one can produce highly nontrivial exactly computable answers for correlators and partition functions (e.g. 2-point correlators given by power laws with special rational exponents, 4-point correlators written in terms of hypergeometric functions and genus 1 partition function written in terms of Jacobi theta functions).

The plan of the class is to develop conformal field theory from scratch – its general structure (operator product expansions, stress-energy tensor, central charge, primary fields and descendants etc.), discuss various mathematical pictures/axiomatizations of CFT (Segal's axioms, vertex algebras) and introduce a zoo of examples:

- Free scalar field.
- Free scalar field with values in the circle of radius r. (This model turns out to be sensitive to whether r^2 is a rational number.)
- Free fermion. (The model corresponding the the famous Ising lattice statistical model.)
- Minimal models of rational conformal field theory.
- Wess-Zumino-Witten model. (This model is deeply related to 3-dimensional Chern-Simons theory – a prototypical topological field theory.)
- Witten's A-model. It has interesting correlators related to enumerative geometry of the target Kähler manifold X. Different aspects of this model lead to notions of quantum cohomology, Gromov-Witten invariants and Fukaya category.

No background in quantum field theory is assumed in the course; the exposition is intended to be self-contained. On the other hand, some background in differential geometry, complex analysis and (a bit of) representation theory is assumed.

Some references:

- Paul Ginsparg,"Applied conformal field theory," hep-th/9108028.
- Toshitake Kohno, "Conformal field theory and topology," Vol. 210 AMS 2002.
- Philippe Di Francesco, Pierre Mathieu, David Sénéchal, "Conformal Field Theory," Springer 1997.