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Multiresolution on compact groups

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Abstract

Given a compact group M, we define the notion of multiresolution of $L^2(M)$ with respect to an infinite sequence of subgroups $G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots$ such that $G = \cup_{k=0}^{\infty} G_k$ is a dense subgroup of M. We give characterizations of various axioms of multiresolution, demonstrate the existence and give the construction of a wavelet basis for $L^2(M)$. We also construct stationary multiresolution and wavelets from cyclic vectors. An example of multiresolution on a non-abelian compact group is given for the infinite dihedral group, or isomorphically the real orthogonal group in dimension two. © 1999 Published by Elsevier Science Inc. All rights reserved.

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0. Introduction

The idea of multiresolution analysis introduced by Mallat [4] has proved to be a fundamental tool in the construction of wavelet, and has been extended in many directions. In [2], Koh et al. examined the multiresolution of $L^2([0, 2\pi))$, the space of square integrable 2π-periodic functions, and proved some very simple yet elegant results. We recall the basic definition below.

For $k = 0, 1, 2, \ldots$, define the operator $T_k : L^2([0, 2\pi)) \to L^2([0, 2\pi))$ by

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$$(T_k f)(x) = f\left(x - \frac{2\pi}{2^k}\right), \quad f \in L^2([0,2\pi)), \quad x \in [0,2\pi).$$

A sequence of subspaces $\{V_k : k \in \mathbb{Z}^+\}$ of $L^2([0,2\pi))$ is called a multiresolution (MR) of $L^2([0,2\pi))$ if the following conditions hold:

MR1: $\dim V_k = 2^k$ and there exists $\phi_k \in V_k$ such that the set $\{T_k^l \phi_k : l = 0, 1, \dots, 2^k - 1\}$ is a basis of V_k for each k,
MR2: $V_k \subseteq V_{k+1}, \forall k = 0, 1, 2, \dots,$
MR3: $\overline{\cup_{k=0}^{\infty} V_k} = L^2([0,2\pi))$.

Note that the set $\{T_k^l : l = 0, 1, \dots, 2^k - 1\}$ forms a group of unitary operators of $L^2([0,2\pi))$, and is isomorphic to $\mathbb{Z}_{2^k}$, the cyclic group of order 2^k, or isomorphically the group of 2^kth roots of unity.

This paper is an attempt to extend the notion of multiresolution and to study their basic properties in a different direction, and is based on the observation that results in [2] are essentially group theoretical in nature, and can be better understood in the context of compact groups. To this end, we shall need to employ Peter–Weyl theory of representations of compact groups and operator Fourier transforms in the place of traditional Fourier series analysis.

1. Multiresolution of $L^2(M)$ with respect to a MR-group sequence

Throughout this paper, let M be a compact group. We normalize the Haar measure on M so that the total volume of M is 1. Denote $L^2(M)$ the space of L^2 functions of M with respect to the Haar measure.

Definition 1.1 (*MR-group sequence*). A sequence of finite subgroups of M, $\{G_k\}_{k=0}^{\infty}$ is called an MR-group sequence of M if the following conditions are satisfied:
(1) $G_k \subseteq G_{k+1}, \forall k = 0, 1, \dots,$
(2) $G = \cup_{k=0}^{\infty} G_k$ is a dense subgroup of M.

Examples. (a) Let T be the one dimensional torus:

$$T = \{t \in \mathbb{C}^{\times} : |t| = 1\},$$

and $G_k = \{t \in \mathbb{C}^{\times} | t^{2^k} = 1\} = \{e^{i2\pi l/2^k} | l = 0, 1, \dots, 2^k - 1\}$. Then $\{G_k\}_{k=0}^{\infty}$ is an MR-group of sequence of T. More generally let $\mathscr{U}_k = \{t \in \mathbb{C}^{\times} | t^k = 1\}$. Then $\{\mathscr{U}_{m_k}\}_{k=0}^{\infty}$ is an MR-group sequence of T if and only if $m_k | m_{k+1}$, $\forall k$, and $m_k \to \infty$ as $k \to \infty$.

(b) Fix any compact abelian group A and an MR-group sequence $\{G_k\}_{k=0}^{\infty}$ of A. Given an action of a finite group B on A, namely a homomorphism

$b \to \sigma(b)$ of B into the group of automorphisms of A, define the semi-direct product of A and B with respect to the action σ as follows:

$$A \rtimes_\sigma B = \{(a, b) \mid a \in A, b \in B\}$$

with the group law

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1\sigma(b_1)(a_2), b_1b_2).$$

Suppose the action σ preserves each G_k, then we may form the subgroups $G_k \rtimes_\sigma B$. Clearly $\{G_k \rtimes_\sigma B\}_{k=0}^{\infty}$ is an MR-group sequence of $A \rtimes_\sigma B$.

Let $L : M \to U(L^2(M))$ be the left regular representation of M in $L^2(M)$:

$$(L(g)f)(x) = f(g^{-1}x), \quad f \in L^2(M), \quad g, x \in M,$$

where $U(L^2(M))$ denotes the group of unitary operators on $L^2(M)$.

Definition 1.2 (*Multiresolution of $L^2(M)$ with respect to $\{G_k\}_{k=0}^{\infty}$*). Let $\{G_k\}_{k=0}^{\infty}$ be an MR-group sequence of M. Let $V_k, k = 0, 1, \dots,$ be a sequence of finite dimensional subspaces of $L^2(M)$. Then $\{V_k\}_{k=0}^{\infty}$ is called a multiresolution of $L^2(M)$ with respect to $\{G_k\}_{k=0}^{\infty}$ if the following conditions hold:

MR1: dim $V_k = |G_k|$ and there exists $\phi_k \in V_k$ such that the set $\{L(G_k)\phi_k\}$ is a basis of V_k for each k,

MR2: $V_k \subseteq V_{k+1}, \forall k = 0, 1, \dots,$

MR3: $\overline{\cup_{k=0}^{\infty} V_k} = L^2(M)$.

1.1. Characterization of MR1

Let H be a finite subgroup of M and ϕ be a representation of H in a Hilbert space $\mathscr{H}$. Let $u_1, \dots, u_m \in \mathscr{H}$ and $S = \{\phi(H)u_i\}_{1 \leqslant i \leqslant m}$. Consider the H-invariant space $V = \text{span } S$.

Denote L the left regular representation of H.

Proposition 1.1.1. *The followings are equivalent*:

(i) S *is linearly independent*.

(ii) $\phi|V \cong \underbrace{L \oplus \cdots \oplus L}_{m}$, *the direct sum of m copies of the representation L*.

Proof. Suppose S is linearly independent. Let $\mathbb{C}[H]$ be the group algebra of H, where H acts by left translation. Recall that for a finite H, this is another realization of the left regular representation of H. Define a map

$$T : \underbrace{\mathbb{C}[H] \oplus \cdots \oplus \mathbb{C}[H]}_{m} \to V$$

as follows: $T((h_1,\ldots,h_m)) = \phi(h_1)u_1 + \cdots + \phi(h_m)u_m$ for $h_1,\ldots,h_m \in H$, and extend T multi-linearly. T is clearly surjective and H-intertwining. T is easily seen to be injective since S is linearly independent. Thus

$$\phi|_V \cong \underbrace{\mathbb{C}[H] \oplus \cdots \oplus \mathbb{C}[H]}_{m} \cong \underbrace{L \oplus \cdots \oplus L}_{m}.$$

Note that $\dim V \leqslant m|H|$. However (ii) implies $\dim V = m|H|$ so S must be linearly independent. $\square$

We now consider the following more general setting, but with the number of vectors $m = 1$:

Let K be a compact group and ϕ be a unitary representation of K in a Hilbert space $\mathscr{H}$ with the inner product $\langle , \rangle$. Let u be a non-zero vector in $\mathscr{H}$. Consider the K-invariant closed subspace $V = \overline{\mathrm{span}\{\phi(K)u\}}$.

Question. When is $\phi|_V \cong$ Left regular representation of K?

The next proposition is quite standard. We give a proof for the sake of completeness.

Proposition 1.1.2. *Define the map* $T : V \to L^2(K)$ *by* $T(v)(k) = \langle v, \phi(k)u\rangle$, $k \in K$. *Then*

(i) *T is a K-equivariant imbedding.*

(ii) $T(V) = \overline{\mathrm{span}\{L(K)f_u\}}$, *where* $f_u = T(u) \in L^2(K)$ *is given by* $f_u(k) = \langle u, \phi(k)u\rangle$, $k \in K$.

(iii) *$\phi|_V$ is contained in the left regular representation of K.*

Proof. One can easily check T is a K-map. To see T is injective, let $T(v) = 0$ for some $v \in V$. Then $\langle v, \phi(k)u\rangle = 0$ for all k in K. Thus $v \in V^\perp$, the orthogonal complement of V in $\mathscr{H}$. Since $v \in V$, v must be zero.

Since V is the closed span of $\phi(k)u$, where $k \in K$, and since T is a K-equivariant imbedding, we see $T(V)$ must be the closed span of $L(K)T(u)$. We thus have (ii).

(iii) follows directly from (i). $\square$

Definition 1.1.3 (*Cyclic vectors of* $L^2(K)$). Let $f \in L^2(K)$. We say f is a cyclic vector of $L^2(K)$ if $\overline{\mathrm{span}\{L(K)f\}} = L^2(K)$.

From Proposition 1.1.2, we see that $\phi|_V \cong$ Left regular representation of K, if and only if f_u is a cyclic vector of $L^2(K)$, where $f_u(k) = \langle u, \phi(k)u\rangle$, $k \in K$.

For $f \in L^1(K)$. Define the operator C_f in $\mathrm{End}(L^2(K))$, the algebra of linear operators of $L^2(K)$, by

$$C_f = \int_K f(k)L(k)\ \mathrm{d}k,$$

where $\mathrm{d}k$ is the normalized Haar measure on K. Evidently C_f is the operator of left convolution by f on $L^2(K)$. Namely $C_f(h)(x) = (f * h)(x) = \int_K f(y)h(y^{-1}x)\ \mathrm{d}y = \int_K f(xy^{-1})h(y)\ \mathrm{d}y$, where $h \in L^2(K)$. It is a general result in real analysis that $f * h \in L^2(K)$ if $f \in L^1(K), h \in L^2(K)$. Thus C_f is a well-defined operator for $f \in L^1(K)$, and in particular for $f \in L^2(K)$ since $L^2(K) \subset L^1(K)$.

For $f_1, f_2 \in L^2(K)$, let $(f_1, f_2) = \int_K f_1(k)\overline{f_2(k)}\ \mathrm{d}k$. Also let $\|h\|_{L^2} = (h,h)^{(1/2)}$ and $h^*(x) = \overline{h(x^{-1})}$, for $h \in L^2(K)$. The following lemma is simple but important for our purpose.

Lemma 1.1.4. *Let $f, h \in L^2(K)$. Then*

(i) $[C_f(h)](k) = (L(k^{-1})f, h^*)$ *for all $k \in K$.*

(ii) $C_f(h) = 0$ *if and only if h^* is orthogonal to the span of K-translates of f.*

Proof. (i) follows from a straightforward computation and (ii) follows directly from (i). □

Definition 1.1.5 (*Non-commutative Fourier transform* [1]). Denote by $\hat{K}$ the set of equivalence classes of finite dimensional continuous irreducible unitary representations of K. For $f \in L^1(K)$ and $\pi \in \hat{K}$, define the operator in $\mathrm{End}(\pi)$ by $\hat{f}(\pi) = \int_K f(k)\pi(k)\ \mathrm{d}k$. $\hat{f}(\pi)$ is called the operator-Fourier coefficient of f at π, or simply the Fourier coefficient of f at π.

Let

$$L^2(K) = \bigoplus_{\pi \in \hat{K}} L(\pi)$$

be the isotypic decomposition of the left regular representation of K. According to the Peter–Weyl theorem, we have

$$L(\pi) \cong \underbrace{\pi \oplus \cdots \oplus \pi}_{d(\pi)},$$

where $d(\pi) = \dim(\pi)$. Each $L(\pi)$ is preserved by the left convolution operator C_f. We let $C_{f,\pi} = C_f|_{L(\pi)}$.

Theorem 1.1.6. *Let $f \in L^2(K)$. Then the followings are equivalent*:

(i) *f is cyclic.*

(ii) *The operator of left convolution by f, $C_f(h) = f * h$ on $L^2(K)$ is bijective.*

(iii) $C_{f,\pi} : L(\pi) \to L(\pi)$ *is invertible for all $\pi \in \hat{K}$.*

(iv) $\hat{f}(\pi) : \pi \to \pi$ *is invertible for all $\pi \in \hat{K}$.*

Remark. IN (ii), the inverse of C_f may not be bounded. We also note that the above implies that the set of cyclic vectors in $L^2(K)$ is closed under convolution.

Proof. Denote $V = \overline{\text{span}\{L(K)f\}}$, as before.

(i) $\Rightarrow$ (ii). Suppose $V = L^2(K)$. Let h be a function in $L^2(K)$ such that $C_f(h) = 0$. By Lemma 1.1.4, h^* is orthogonal to the K-translates of f and it will thus be orthogonal to V. Since $V = L^2(K)$, we see that $h^* = 0$. Therefore $h = 0$ which implies C_f is injective.

Furthermore C_f preserves $L(\pi)$. We claim that $C_f(L(\pi)) = L(\pi)$ for all $\pi \in \hat{K}$, namely C_f is surjective on $L(\pi)$. This is because C_f is an injective map on $L(\pi)$ and $L(\pi)$ is finite dimensional. Thus $L(\pi)$ is contained in the image of C_f for all π. Since $L^2(K) = \oplus_{\pi\in\hat{K}} L(\pi)$, we see C_f is surjective.

(ii) $\Rightarrow$ (i). Suppose C_f is bijective. Let $h \in V^\perp$. Then $C_f(h^*) = 0$ again by Lemma 1.1.4. Since C_f is injective, we have $h^* = 0$ which implies $h = 0$. Combining this with the fact that V is closed, we see $V = L^2(K)$.

(ii) $\Rightarrow$ (iii). Suppose C_f is bijective. For each $\pi \in \hat{K}, C_{f,\pi} = C_f|_{L(\pi)}$ is clearly injective. Since dim $L(\pi)$ is finite, $C_{f,\pi}$ is invertible.

(iii) $\Rightarrow$ (ii). Suppose $C_{f,\pi}$ is invertible for all $\pi \in \hat{K}$. Let $C_f(h) = 0$ for some $h \in L^2(K)$. Let $h = \sum_{\pi\in\hat{K}} h_\pi$ be the isotypic decomposition of h. Then for each $\pi, C_{f,\pi}(h_\pi) = C_f(h_\pi) = 0$. Since $C_{f,\pi}$ is invertible in $L(\pi)$, we have $h_\pi = 0$. Thus $h = 0$. Surjectivity of C_f also follows easily from the surjectivity of $C_{f,\pi}$ for all $\pi \in \hat{K}$, as in the proof of (i) $\Rightarrow$ (ii).

(iii) $\iff$ (iv). We have $L^2(K) = \sum_{\pi\in\hat{K}} L(\pi)$ and by Peter–Weyl theorem,

$$L(\pi) \cong \underbrace{\pi \oplus \cdots \oplus \pi}_{d(\pi)}.$$

Under this identification, we have

$$C_{f,\pi} \cong \underbrace{\hat{f}(\pi) \oplus \cdots \oplus \hat{f}(\pi)}_{d(\pi)}.$$

Thus $C_{f,\pi}$ is invertible if and only if $\hat{f}(\pi)$ is invertible. $\square$

Corollary 1.1.7. *Let ϕ be a unitary representation of K in a Hilbert space $\mathscr{H}$, $u \in \mathscr{H}$. Denote $V = \overline{\text{span}\{\phi(K)u\}}$. Then the following are equivalent*:

(i) $\phi|_V \cong$ *Left regular representation of K,*

(ii) f_u *is a cyclic in $L^2(K)$, where $f_u(k) = \langle u, \phi(k)u\rangle, k \in K$,*

(iii) C_{f_u} = *the operator of left convolution by f_u, is bijective on $L^2(K)$,*

(iv) $\hat{f_u}(\phi) = \int_K \langle u, \phi(k)u\rangle \pi(k)\, dk$ *is an invertible operator in the representation space of π for all $\pi \in \hat{K}$.*

In fact we can give a stronger but equivalent condition of (iv). To do this, we pause to prove a general result for compact groups, which can be viewed as an analog of Bochner's theorem on positive definite functions.

Proposition 1.1.8. *Let ϕ be a unitary representation of K in a Hilbert space $\mathscr{H}$, and $u_1, \dots, u_m \in \mathscr{H}$. Define $f_{i,j} \in L^2(K)$ by*

$$f_{i,j}(k) = \langle u_i, \phi(k)u_j \rangle, \quad k \in K, 1 \leqslant i, j \leqslant m.$$

(a) *Then for any $\pi \in \hat{K}$, and any vectors $w_1, \dots, w_m$ in the representation space of π, we have*

$$\sum_{1 \leqslant i,j \leqslant m} \langle \hat{f}_{i,j}(\pi)w_j, w_i \rangle \geqslant 0.$$

(b) *Further assume that $K = H$, a finite group and $\{\phi(H)u_i\}_{1 \leqslant i \leqslant m}$ is linearly independent. Then*

$$\sum_{1 \leqslant i,j \leqslant m} \langle \hat{f}_{ij}(\pi)w_j, w_i \rangle = 0$$

if and only if $w_i = 0$ for $1 \leqslant i \leqslant m$.

Proof. We have

$$\begin{aligned}
\sum_{1 \leqslant i,j \leqslant m} \langle \hat{f}_{ij}(\pi)w_j, w_i \rangle &= \sum_{i,j} \int_K \langle u_i, \phi(x)u_j \rangle \langle \pi(x)w_j, w_i \rangle \, \mathrm{d}x \\
&= \sum_{i,j} \int_K \int_K \langle u_i, \phi(x^{-1}y)u_j \rangle \langle \pi(x^{-1}y)w_j, w_i \rangle \, \mathrm{d}x \, \mathrm{d}y \\
&= \sum_{i,j} \int_K \int_K \langle \phi(x)u_i, \phi(y)u_j \rangle \langle \pi(y)w_j, \pi(x)w_i \rangle \, \mathrm{d}x \, \mathrm{d}y.
\end{aligned}$$

Here $\langle , \rangle$ denotes inner product in the appropriate space.

Fix a K equivariant isometric imbedding:

$$T : \pi \hookrightarrow L^2(K)$$

and let $b_i = T(w_i) \in L^2(K)$. Then

$$\langle \pi(y)w_j, \pi(x)w_i \rangle = (L(y)b_j, L(x)b_i) = \int_K b_j(y^{-1}z)\overline{b_i(x^{-1}z)} \, \mathrm{d}z.$$

Therefore

$$\begin{aligned}
\sum_{1 \leqslant i,j \leqslant m} \langle \hat{f}_{ij}(\pi)w_j, w_i \rangle &= \sum_{i,j} \int_K \int_K \int_K \langle \phi(x)u_i, \phi(y)u_j \rangle b_j(y^{-1}z)\overline{b_i(x^{-1}z)} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\
&= \sum_{i,j} \int_K \int_K \int_K \langle \overline{b_i(x^{-1}z)}\phi(x)u_i, \overline{b_j(y^{-1}z)}\phi(y)u_j \rangle \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\
&= \int_K \langle c(z), c(z) \rangle \, \mathrm{d}z \geqslant 0,
\end{aligned}$$

where $c(z) = \sum_{1 \leqslant i \leqslant m} \int_K \overline{b_i(x^{-1}z)}\phi(x)u_i \, \mathrm{d}x$. This proves (a).

We now prove part (b). Recall the normalize Haar measure for a finite group is the counting measure divided by the order of the group. Thus

$$\sum_{1 \leqslant i,j \leqslant m} \langle \hat{f}_{ij}(\pi) w_j, w_i \rangle = 0$$

implies that

$$c(z) = (1/H) \sum_{1 \leqslant i \leqslant m} \sum_{x \in H} \overline{b_i(x^{-1}z)} \phi(x) u_i = 0$$

for any $z \in H$. Since $\{\phi(H)u_i\}_{1 \leqslant i \leqslant m}$ is linearly independent, we see that $b_i(x^{-1}z) = 0$ for $x, z \in H$ and $1 \leqslant i \leqslant m$. Thus $b_i = 0$ for any i. Since T is an imbedding and $b_i = T(w_i)$, we conclude that $w_i = 0$ for $1 \leqslant i \leqslant m$. $\square$

Corollary 1.1.7 together with Proposition 1.1.8 ($m = 1$) imply the following corollaries.

Corollary 1.1.9. *Let ϕ be a unitary representation of K in a Hilbert space $\mathscr{H}, u \in \mathscr{H}$. Denote $V = \overline{\operatorname{span}\{\pi(K)u\}}$. Then the followings are equivalent*:

(i) $\phi|_V \cong$ *Left regular representation of* K,

(ii) f_u *is cyclic in* $L^2(K)$, *where* $f_u(k) = \langle u, \phi(k)u \rangle,\ k \in K$,

(iii) C_{f_u} $=$ *the operator of left convolution by* f_u, *is bijective on* $L^2(K)$,

(iv) $\hat{f}_u(\pi) = \int_K \langle u, \phi(k)u \rangle \pi(k)\, \mathrm{d}k$ *is a positive definite operator in the representation space of* π *for all* $\pi \in \hat{K}$.

The above results for compact groups will hold in particular for finite groups. In view of Proposition 1.1.1, we obtain the following corollary.

Corollary 1.1.10 (Characterization of MR1). *Notations as in Definition* 1.2. *The followings are equivalent*:

(i) $\{V_k\}_{k=0}^{\infty}$ *satisfies MR*1.

(ii) *For all* $k = 0, 1, \ldots,$ *there exists* $\phi_k \in V_k$ *such that the function* $f_{\phi_k} \in L^2(G_k)$ *is cyclic, where* $f_{\phi_k}(g) = (\phi_k, L(g)\phi_k)$, *for* $g \in G_k$.

(iii) *For all* $k = 0, 1, \ldots,$ *there exists* $\phi_k \in V_k$ *such that the operator-Fourier coefficient* $\hat{f}_{\phi_k}(\pi) = (1/|G_k|) \sum_{g \in G_k} (\phi_k, L(g)\phi_k) \pi(g)$ *is positive definite in the representation space of* π *for all* $\pi \in \hat{G}_k$.

Remark. For $M = T,\ G_k = \{\mathrm{e}^{\mathrm{i}2\pi l/2^k} | l = 0, 1, \ldots, 2^k - 1\}$, (iii) states that for each k, there exists $\phi_k \in V_k$ such that for all $j = 0, 1, \ldots, 2^k - 1$, $\sum_{l=0}^{2^k-1} (\phi_k, T_k^l \phi_k) \mathrm{e}^{\mathrm{i}2\pi jl/2^k} > 0$. This is first proved in [2].

1.2. Characterization of MR2

Definition 1.2.1. For any $\pi \in \hat{M}$, fix an orthonormal basis in the representation space of π, denoted by $\{e_i;\ i = 1, \ldots, d(\pi)\}$. For $1 \leqslant i, j \leqslant d(\pi)$, define $\pi_{ij} \in L^2(M)$ by

$$\pi_{ij}(x) = \langle e_i, \pi(x)\ e_j \rangle, \quad x \in M.$$

We list below some well-known properties of these matrix coefficients.

Lemma 1.2.2 [1, p. 129].

(i) *For each $\pi \in \hat{M}$, the collection $\{d(\pi)^{1/2}\pi_{ij} \mid 1 \leqslant i, j \leqslant d(\pi)\}$ form an orthonormal basis of $L(\pi)$.*

(ii) *The collection $\{d(\pi)^{1/2}\pi_{ij} \mid \pi \in \hat{M}, 1 \leqslant i, j \leqslant d(\pi)\}$ form a complete orthonormal basis of $L^2(M)$.*

(iii) *$L(\pi) * L(\sigma) = 0$ if $\pi, \sigma \in \hat{M}$ and $\pi \ncong \sigma$.*

(iv) *$\pi_{ij} * \pi_{kl} = (1/d(\pi))\delta_{jk}\pi_{il}$, where δ_{jk} is the Kronicker symbol.*

Theorem 1.2.3. *Let H be a finite subgroup of M. Let $\alpha = \sum_{\pi \in \hat{M}, 1 \leqslant i,j \leqslant d(\pi)} a_{\pi_{ij}}\pi_{ij}$ and $\beta = \sum_{\pi \in \hat{M}, 1 \leqslant i,j \leqslant d(\pi)} b_{\pi_{ij}}\pi_{ij}$ be two functions in $L^2(M)$. Then the followings are equivalent*:

(i) $\alpha \in \operatorname{span}\{L(H)\beta\}$,

(ii) *There exists a function $c \in L^2(H)$ such that for all $\pi \in \hat{M}$, and $1 \leqslant i, j \leqslant d(\pi)$ we have $a_{\pi_{ij}} = (c, \eta_{\pi_{ij}})_H$, where $(,)_H$ denotes the inner product in $L^2(H)$ given by*

$$(\phi_1, \phi_2)_H = \frac{1}{|H|} \sum_{h \in H} \phi_1(h)\overline{\phi_2(h)},$$

and $\eta_{\pi_{ij}} \in L^2(H)$ is given by $\eta_{\pi_{ij}} = \sum_{l-1}^{d(\pi)} \overline{b_{\pi_{ij}}}\pi_{il}|H$.

Remark. If $M = T,\ H = \{e^{i2\pi l/2^k} | l = 0, 1, \ldots, 2^k - 1\}$, and write

$$\alpha = \sum_{n \in \mathbb{Z}} \hat{\alpha}(n)\ e^{inx}, \qquad \beta = \sum_{n \in \mathbb{Z}} \hat{\beta}(n)\ e^{inx},$$

then (iii) can be stated as follows: There exists $\{c(j)\}_{j=0}^{2^k-1}$ such that

$$\hat{\alpha}(n) = \frac{1}{2^k} \sum_{j=0}^{2^k-1} c(j)\hat{\beta}(n)\ e^{-i2\pi jn/2^k} = c_n\hat{\beta}(n),$$

where

$$c_n = \frac{1}{2^k} \sum_{j=0}^{2^k-1} c(j)\ e^{-i2\pi jn/2^k}.$$

Clearly this is the same as $\hat{\alpha}(n) = c_n \hat{\beta}(n)$ with $\{c_n\}_{n\in\mathbb{Z}}$ being a 2^k-periodic sequence. This was first proved in [2].

Proof. Let (π, W) be an irreducible unitary representation of M. Fix an orthonormal basis $\{e_i;\ i = 1, \ldots, d(\pi)\}$ of W as in Definition 1.2.1. Note that $\hat{\alpha}(\pi) \in \mathrm{End}(W)$. The (i,j) entry of $\hat{\alpha}(\pi)$ is given by

$$(\hat{\alpha}(\pi)\ e_j,\ e_i) = \int_M \alpha(x)(\pi(x)\ e_j,\ e_i)\ \mathrm{d}x = \int_M \alpha(x)\overline{\pi_{ij}(x)}\ \mathrm{d}x = \frac{a_{\pi_{ij}}}{d(\pi)}.$$

In particular,

$$\hat{\alpha}(\pi)\ e_j = \sum_{l=1}^{d(\pi)} \frac{a_{\pi_{lj}}}{d(\pi)}\ e_l.$$

Note that $\alpha \in \mathrm{span}\{L(H)\beta\}$ if and only if there exists $c \in L^2(H)$ such that

$$\alpha = \frac{1}{|H|} \sum_{h\in H} c(h)L(h)\beta.$$

By the uniqueness of operator Fourier transform, this is equivalent to $\hat{\alpha}(\pi) = (1/|H|)\sum_{h\in H} c(h)(L(h)\beta)^{\wedge}(\pi)$ for all $\pi \in \hat{M}$. The latter is equal to

$$\left[\frac{1}{|H|}\sum_{h\in H} c(h)\pi(h)\right] \circ \hat{\beta}(\pi) = (\pi_c \circ \hat{\beta})(\pi),$$

where $\pi_c = \frac{1}{|H|}\sum_{h\in H} c(h)\pi(h) \in \mathrm{End}(U)$.

We compute the (i,j) entry of $(\pi_c \circ \hat{\beta})(\pi)$:

$$\begin{aligned}
(\pi_c \circ \hat{\beta}(\pi)e_j,\ e_i) &= \left(\pi_c\left(\sum_{l=1}^{d(\pi)} \frac{b_{\pi_{lj}}}{d(\pi)}\ e_l\right),\ e_i\right) = \frac{1}{d(\pi)}\sum_{l=1}^{d(\pi)} b_{\pi_{lj}}(\pi_c(e_l),\ e_i) \\
&= \frac{1}{d(\pi)}\sum_{l=1}^{d(\pi)} b_{\pi lj}\frac{1}{|H|}\sum_{h\in H} c(h)(\pi(h)\ e_l,\ e_i) \\
&= \frac{1}{d(\pi)}\frac{1}{|H|}\sum_{h\in H} c(h)[\sum_{l=1}^{d(\pi)} b_{\pi_{lj}}\overline{\pi_{il}(h)}] = \frac{1}{d(\pi)}(c, \eta_{\pi_{ij}})H.
\end{aligned}$$

Comparing the (i,j) entries of $\hat{\alpha}(\pi)$ and $(\pi_c \circ \hat{\beta})(\pi)$, the result follows. $\square$

We can now characterize MR2. Observe that $V_k \subseteq V_{k+1}$ if and only if $\phi_k \in \mathrm{span}\{L(G_{k+1})\phi_{k+1}\}$. Thus we have the following corollary.

Corollary 1.2.4 (Characterization of MR2). *Suppose* $\{V_k\}_{k=0}^{\infty}$ *is a sequence of subspaces of* $L^2(M)$ *satisfying* MR1. *Write each* $\phi_k = \sum \phi_{\pi_{ij}}(k)\pi_{ij}$ *for* $k = 0, 1, \ldots,$ *where* π_{ij} *is as in Definition* 1.2.2. *Then* $\{V_k\}_{k=0}^{\infty}$ *satisfies* MR2 *if and* *only if for all $k = 0, 1, \ldots,$ there exists a function $c_k \in L^2(G_{k+1})$ such that for all $\pi \in \hat{M}$, and $1 \leqslant i, j \leqslant d(\pi)$ we have $\phi_{\pi_{ij}}(k) = (c_k, \eta_{\pi_{ij}}(k))_{G_{k+1}}$, where*

$$\eta_{\pi_{ij}}(k) = \sum_{l=1}^{d(\pi)} \overline{\phi_{\pi_{lj}}(k+1)}\pi_{il}|_{G_{k+1}}.$$

1.3. Characterization of MR3

Let G be a dense subgroup of M.

Theorem 1.3.1. *Let V be a G-invariant subspace of $L^2(M)$. Define the set*

$$\Omega = \bigcap_{f \in V} \{\pi \in \hat{M} | \hat{f}(\pi) \text{ is singular}\}.$$

Then the followings are equivalent:
(i) $\overline{V} = L^2(M)$.
(ii) $\Omega = \emptyset$, *the empty set.*

Remarks. (a) If $V = \mathrm{span}\{L(G)f\}$ where $f \in L^2(M)$, then the result of this theorem reduces to the characterization of cyclic vectors. See Theorem 1.1.6.

(b) The theorem can be viewed as a compact group analog of a result of Wiener on translation invariant subspaces of $L^2(\mathbb{R})$, see [5].

Proof. (i) $\Rightarrow$ (ii). Suppose $\Omega \neq \emptyset$. Then there exists $\pi \in \hat{M}$ such that $\hat{f}(\pi)$ is singular for all f in V. Observe that the operator norm of $\hat{f}(\pi)$ is less than or equal to $|f|_{L^1} \leqslant |f|_{L^2}$, and so the map from $L_2(M)$ to End (π) given by $f \mapsto \hat{f}(\pi)$ is continuous. Thus the set of f such that $\hat{f}(\pi)$ is singular is a closed set of $L^2(G)$, and so $\hat{f}(\pi)$ is singular for all $f \in \bar{V} = L^2(M)$. Take $f_0 = d(\pi)\bar{\chi}_\pi$, where χ_π is the character of the representation π. Then a simple computation shows that $\hat{f}_0(\pi)$ is the identity operator in the representation space of π. We thus have contradiction.

(ii) $\Rightarrow$ (i). Suppose $\bar{V} \neq L^2(M)$. Then there exists a non-zero function $h \in L^2(M)$ which is orthogonal to V. By Lemma 1.1.4, $C_f(h^*) = 0$ for $f \in V$. Pick $\pi \in \hat{M}$ such that the π-isotypic component h^*_π of h^* is not zero. Since C_f preserves $L(\pi)$, we see $C_f(h^*_\pi) = 0$ and so $\hat{f}(\pi)$ is not invertible for all $f \in V$, that is $\pi \in \Omega$. $\square$

Now let $V_k, k = 0, 1, \ldots,$ be a sequence of finite dimensional subspaces of $L^2(M)$ satisfying MR1. Observe that $(L(x)f)^\wedge(\pi) = \pi(x)\hat{f}(\pi)$ for $x \in M$, $f \in L^2(M)$ and $\pi \in \hat{M}$, we see that $\hat{f}(\pi)$ is singular for all $f \in V_k$ if and only if $\hat{\phi}_k(\pi)$ is singular. Thus if we let $V = \bigcup_{k=0}^\infty V_k$, then we have

$$\bigcap_{f \in V} \{\pi \in \hat{M} | \hat{f}(\pi) \text{ singular}\} = \bigcap_{k=0}^{\infty} \{\pi \in \hat{M} | \hat{\phi}_k(\pi) \text{ is singular}\}.$$

We therefore have the following corollary.

Corollary 1.3.2 (Characterization of MR3). *Assume the notations of Definition* 1.2 *and* $\{V_k\}_{k=0}^{\infty}$ *satisfies* MR1. *Then* $\{V_k\}_{k=0}^{\infty}$ *satisfies* MR3 *if and only if*

$$\Omega = \bigcap_{k=0}^{\infty} \{\pi \in \hat{M} | \hat{\phi}_k(\pi) \textit{ is singular}\} = \emptyset,$$

the empty set.

Remark. In the classical case of $M = T$ and $G_k = \{e^{i2\pi l/2^k} | l = 0, 1, \dots, 2^k - 1\}$, then $\Omega = \bigcap_{k=0}^{\infty} \{n \in \mathbb{Z} | \hat{\phi}_k(n) = 0\}$. The result again is first proved in [2].

2. Existence and construction of orthonormal wavelets

Let $\{G_k\}_{k=0}^{\infty}$ be an MR-group sequence of M, and $\{V_k\}_{k=0}^{\infty}$ be a multiresolution of $L^2(M)$ with respect to $\{G_k\}_{k=0}^{\infty}$

Definition 2.1. A set of functions

$$\bigcup_{k=0}^{\infty} \bigcup_{l=1}^{\alpha_k} \{\phi_k^l\}$$

in $L^2(M)$ is called orthonormal wavelets if

$$\bigcup_{k=0}^{\infty} \bigcup_{l=1}^{\alpha_k} \{L(G_k)\phi_k^l\}$$

is an orthonormal basis of $L^2(M)$.

For $k = 0, 1, \dots,$ let W_k be the orthogonal complement of V_k in V_{k+1} with respect to the standard inner product of $L^2(M)$. Since G_k preserves the inner product, W_k is also a G_k module.

By MR3, we have

$$L^2(M) = V_0 \oplus \left(\bigoplus_{k=0}^{\infty} W_k \right).$$

Set $t_k = |G_{k+1}/G_k|, k \geqslant 0$.

G_k acts on V_k and V_{k+1} via the restriction of the left regular representation of M. From MR1, we see that $V_k \cong L^2(G_k)$ as G_k modules. Furthermore since $V_{k+1} \cong L^2(G_{k+1})$ as G_{k+1} modules, we have

$$V_{k+1} \cong \underbrace{L^2(G_k) \oplus \cdots \oplus L^2(G_k)}_{t_k}, \quad \text{and} \quad W_k \cong \underbrace{L^2(G_k) \oplus \cdots \oplus L^2(G_k)}_{t_k - 1}$$

as representations of G_k ([6], p. 28 on induced representations).

Let

$$T : \underbrace{L^2(G_k) \oplus \cdots \oplus L^2(G_k)}_{t_k - 1} \to W_k$$

be a unitary G_k-equivariant isomorphism. Denote $\delta_e^1 = (0, \ldots, 0, \delta_e, 0, \ldots, 0)$, where $\delta_e \in L^2(G_k)$ is in the lth component and is defined by $\delta_e(x) = 1$ for $x = e$ and zero otherwise. Set $\phi_k^l = T(\delta_e^l)$, where $1 \leqslant l \leqslant t_k - 1$. Then $\cup_{l=1}^{t_k-1}\{L(G_k)\phi_k^l\}$ is an orthonormal basis of W_k. Similarly from $V_0 \cong L^2(G_0)$, we can find $\phi_0^0 \in V_0$ such that $\{L(G_0)\phi_0^0)\}$ is an orthonormal basis of V_0.

Thus

$$\{L(G_0)(\phi_0^0)\} \bigcup_{k=0}^{\infty} \bigcup_{l=1}^{t_k-1} \{L(G_k)\phi_k^l\}$$

is an orthonormal basis of $L^2(M)$, and so the set

$$\{\phi_0^0\} \bigcup_{k=0}^{\infty} \bigcup_{l=1}^{t_k-1} \{\phi_k^l\}$$

gives orthonormal wavelets of $L^2(M)$.

2.1. Construction of orthonormal wavelets

Let H be a finite group. Let ϕ be a unitary representation of H in a Hilbert space $\mathscr{H}$. Fix a finite set of vectors $u_1, \ldots, u_m \in \mathscr{H}$ and let $V = \operatorname{span}\{\phi(H)u_i\}_{1 \leqslant i \leqslant m}$. Given a set of functions $\{c_{ij}\}_{1 \leqslant i,j \leqslant m} \in L^2(H)$, define

$$v_i = \frac{1}{|H|} \sum_{1 \leqslant j \leqslant m} \sum_{h \in H} c_{ij}(h)\phi(h)u_j.$$

Recall for $1 \leqslant i,j \leqslant m$ the associated functions $f_{ij} \in L^2(H)$ given by $f_{ij}(h) = \langle u_i, \phi(h)u_j \rangle$, $h \in H$. Similarly define $g_{ij} \in L^2(H)$ by $g_{ij}(h) = \langle v_i, \phi(h)v_j \rangle$, $h \in H$. The purpose of this section is to construct explicitly $\{v_i\}_{1 \leqslant i \leqslant m}$ such that $\{\phi(H)v_i\}_{1 \leqslant i \leqslant m}$ is an orthonormal basis of V, given a basis $\{\phi(H)u_i\}_{1 \leqslant i \leqslant m}$ of V.

Recall also for any $f \in L^2(H)$, the operator Fourier transform $\hat{f}(\pi)$ at $\pi \in \hat{H}$ defined by

$$\hat{f}(\pi) = \frac{1}{|H|} \sum_{h \in H} f(h)\pi(h) \in \operatorname{End}(\pi).$$

For any set of linear operators $\{a_{ij}\}_{1 \leqslant i,j \leqslant m}$ in a vector space W, denote $A = (a_{ij})_{1 \leqslant i,j \leqslant m}$ the corresponding matrix. We may view A as an operator in $mW = \underbrace{W \oplus \cdots \oplus W}_{m}$ by the following formula:

$$A((w_1, \ldots, w_m)) = \left(\sum_{j=1}^{m} a_{1j} w_j, \ldots, \sum_{j=1}^{m} a_{mj} w_j \right).$$

Denote $\hat{F}(\pi) = (\hat{f}_{ij}(\pi))_{1 \leqslant i,j \leqslant m}$, $\hat{G}(\pi) = (\hat{g}_{ij}(\pi))_{1 \leqslant i,j \leqslant m}$, and $\hat{C}(\pi) = (\hat{c}_{ij}(\pi))_{1 \leqslant i,j \leqslant m}$.

Lemma 2.1.1. *We have $\hat{G}(\pi) = \hat{C}(\pi)\hat{F}(\pi)\hat{C}(\pi)^{*t}$, namely*

$$\hat{g}_{ij}(\pi) = \sum_{1 \leqslant k,l \leqslant m} \hat{c}_{ik}(\pi)\hat{f}_{kl}(\pi)\hat{c}_{jl}(\pi)^*,$$

for any $\pi \in \hat{H}$, and a^* denotes the adjoint operator of $a \in \operatorname{End}(\pi)$.

Proof. We compute for $x \in H$,

$$\begin{aligned} g_{ij}(x) = \langle v_i, \phi(x) v_j \rangle &= \frac{1}{|H|^2} \sum_{y,z \in H} c_{ik}(y)\overline{c_{jl}(z)} \langle \phi(y) u_k, \phi(xz) u_l \rangle \\ &= \frac{1}{|H|^2} \sum_{y,z \in H} c_{ik}(y)\overline{c_{jl}(z)} f_{kl}(y^{-1}xz). \end{aligned}$$

A simple computation shows that the Fourier coefficient of the function $f_{kl}(y^{-1}xz)$ at π is $\pi(y)\hat{f}_{kl}(\pi)\pi(z)^*$, and hence

$$\begin{aligned} \hat{g}_{ij}(\pi) &= \frac{1}{|H|^2} \sum_{k,l} \sum_{y,z \in H} c_{ik}(y)\pi(y)\hat{f}_{kl}(\pi)\overline{c_{jl}(z)}\pi(z)^* \\ &= \hat{c}_{ik}(\pi)\hat{f}_{kl}(\pi)\hat{c}_{jl}(\pi)^*. \qquad \square \end{aligned}$$

Proposition 2.1.2. *$\{\phi(H)v_i\}_{1 \leqslant i \leqslant m}$ is an orthonormal basis of V if and only if*

$$\hat{C}(\pi)\hat{F}(\pi)\hat{C}(\pi)^{*t} = \frac{1}{|H|} \begin{pmatrix} I_{d(\pi)} & 0 & \cdots & 0 \\ 0 & I_{d(\pi)} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & I_{d(\pi)} \end{pmatrix}$$

for all $\pi \in \hat{H}$. Here $I_{d(\pi)}$ is the identity operator in the representation space of π.

Proof. Clearly $\{\phi(H)v_i\}_{1\leqslant i\leqslant m}$ is orthonormal if and only if $g_{ij} = \delta_{ij}\delta_e$, where $\delta_{ij} = 1, 0$ depending on whether i is equal to j, and $\delta_e(x) = 1$ if $x = e$ and zero elsewhere. By the uniqueness of operator Fourier transform, it is in turn equivalent to $\hat{g}_{ij}(\pi) = \delta_{ij}\hat{\delta}_e(\pi) = (1/|H|)\delta_{ij}I_{d(\pi)}$ for all $\pi \in \hat{H}$, namely

$$\hat{G}(\pi) = \frac{1}{|H|}\begin{pmatrix} I_{d(\pi)} & 0 & \cdots & 0 \\ 0 & I_{d(\pi)} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & I_{d(\pi)} \end{pmatrix}.$$

From Lemma 2.1.1, we see that this is equivalent to

$$\hat{C}(\pi)\hat{F}(\pi)\hat{C}(\pi)^{*t} = \frac{1}{|H|}\begin{pmatrix} I_{d(\pi)} & 0 & \cdots & 0 \\ 0 & I_{d(\pi)} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & I_{d(\pi)} \end{pmatrix}$$

for all $\pi \in \hat{H}$. $\square$

Now suppose that $\{\phi(H)u_i\}_{1\leqslant i\leqslant m}$ is linearly independent. From Proposition 1.1.8, we see that $\hat{F}(\pi)$ is positive definite as an operator in the representation space of $m\pi = \underbrace{\pi \oplus \cdots \oplus \pi}_{m}$.

Theorem 2.1.3. *Let H be a finite group, and ϕ be a unitary representation of H in a Hilbert space $\mathscr{H}$. Let $u_1, \ldots, u_m \in \mathscr{H}$ such that $\{\phi(H)u_i\}_{1\leqslant i\leqslant m}$ is linearly independent, and denote $V = \mathrm{span}\{\phi(H)u_j\}_{1\leqslant j\leqslant m}$. For any $\pi \in \hat{H}$, let*

$$P(\pi) = (p_{ij}(\pi))_{1\leqslant i,j\leqslant m} = \frac{1}{\sqrt{|H|}}\hat{F}(\pi)^{-1/2},$$

where $p_{ij}(\pi) \in \mathrm{End}(\pi)$. Let $c_{ij} \in L^2(H)$ be given by

$$c_{ij}(h) = \sum_{\pi\in\hat{H}} d(\pi)\mathrm{tr}[p_{ij}(\pi)\pi(h^{-1})], \quad h \in H,$$

and tr *denotes the trace of an operator. Then the vectors*

$$v_i = \frac{1}{|H|}\sum_{h\in H} c_{ij}(h)\phi(h)u_j$$

generate an orthonormal basis $\{\phi(H)v_j\}_{1\leqslant j\leqslant m}$ for V.

Proof. By our construction,

$$P(\pi)\hat{F}(\pi)P(\pi)^{*t} = \frac{1}{|H|}\begin{pmatrix} I_{d(\pi)} & 0 & \cdots & 0 \\ 0 & I_{d(\pi)} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & I_{d(\pi)} \end{pmatrix}.$$

The Fourier inversion formula [1, p. 131] implies that $\hat{c}_{ij}(\pi) = p_{ij}(\pi)$, for $\pi \in \hat{H}$. Hence we have

$$\hat{C}(\pi)\hat{F}(\pi)\hat{C}(\pi)^{*t} = \frac{1}{|H|}\begin{pmatrix} I_{d(\pi)} & 0 & \cdots & 0 \\ 0 & I_{d(\pi)} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & I_{d(\pi)} \end{pmatrix}.$$

Proposition 2.1.2 then implies the result. □

2.2. Construction of stationary multiresolution and wavelets from cyclic vectors

Recall the notion of cyclic vectors of $L^2(M)$ (Section 1.1). We first give a concrete description of the set of cyclic vectors of $L^2(M)$.

Fix $\pi \in \hat{M}$. Define for each $d(\pi) \times d(\pi)$ invertible matrix $A = (a_{ij})$ the function

$$f_{\pi,A} = \frac{d(\pi)^{1/2}}{|A|} \sum_{i,j=1}^{d(\pi)} a_{ij}\pi_{ij},$$

where

$$|A| = \sqrt{\sum_{i,j=1}^{d(\pi)} |a_{ij}|^2}.$$

Note that $|f_{\pi,A}|_{L^2} = 1$.

Define the following subset of $L^2(M)$:

$$\mathscr{C}(M) = \left\{ \sum_{\pi\in\hat{M}} a_\pi f_{\pi,A};\ a_\pi \neq 0,\ (a_\pi) \in l^2(\hat{M}),\ A \in GL_{d(\pi)}(\mathbb{C}) \right\}.$$

Notice that the summation $\sum_{\pi\in\hat{M}} a_\pi f_{\pi,A}$ converges to some function in $L^2(M)$ by the well-known Riesz–Fischer theorem.

Proposition 2.2.1. *f is cyclic in $L^2(M)$ if and only if $f \in \mathscr{C}(M)$.*

Proof. Write $f = \sum_{\pi\in\hat{M},1\leqslant i,j\leqslant d(\pi)} a_{\pi_{ij}}\pi_{ij}$, Then we have

$$(\hat{f}(\pi)\ e_j, e_i) = \int_M f(x)\overline{\pi_{ij}(x)}\ \mathrm{d}x = \frac{a_{\pi_{ij}}}{d(\pi)}.$$

By Theorem 1.1.6, f is cyclic if and only $\hat{f}(\pi)$ is invertible for all $\pi \in \hat{M}$, and thus it is equivalent to the fact that the matrix $(a_{\pi_{ij}})_{1\leqslant i,j\leqslant d(\pi)}$ is invertible for all $\pi \in \hat{M}$. The proposition clearly follows after a normalization. □

Theorem 2.2.2. *Let f be a cyclic vector of $L^2(M)$. Let H be a finite subgroup of M. Then the representation of H on $V = \mathrm{span}\{L(H)f\}$ is equivalent to the left regular representation of H on $L^2(H)$.*

Proof. By Proposition 1.1.1, it suffices to show that the set $\{L(H)f\}$ is linearly independent. Suppose that there exists $c_h \in \mathbb{C}$ for each $h \in H$ such that $\sum_{h\in H} c_h L(h)f = 0$. Construct an open neighborhood N_e about the identity e of M such that $N_g \cap N_h = \emptyset$ for all distinct g, h in H, where $N_g = \{g^{-1}v';\ v \in N_e\}$. Define the function $q \in L^2(M)$ as follows:

$$q(x) = \begin{cases} \overline{c_h} & \text{if } x \in N_h, \\ 0 & \text{if } x \notin \cup_{h\in H} N_h. \end{cases}$$

Let $f^*(x) = \overline{f(x^{-1})}$, then we have $\hat{f^*}(\pi) = (\hat{f}(\pi))^*$ for any $\pi \in \hat{M}$, where $(\hat{f}(\pi))^*$ denotes the adjoint operator of $\hat{f}(\pi)$ with respect to the inner product in the representation space of π. Thus f^* is also cyclic, and so the operator C_{f^*} is bijective.

We compute

$$\begin{aligned}(C_{f^*}(q))(g) &= \int_M f^*(gx^{-1})q(x)\ \mathrm{d}x \\ &= \sum_{h\in H} \int_{N_h} f^*(gx^{-1})\overline{c_h}\ \mathrm{d}x = \sum_{h\in H} \int_{N_e} f^*(gx^{-1}h)\overline{c_h}\ \mathrm{d}x \\ &= \int_{N_e} \overline{\sum_{h\in H} c_h(L(h)f)(xg^{-1})}\ \mathrm{d}x = 0.\end{aligned}$$

Since C_{f^*} is bijective, we have $q = 0$. Therefore $c_h = 0$ for all $h \in H$. Thus $\{L(H)f\}$ is linearly independent. □

Theorem 2.2.3. *Let f be a cyclic vector in $L^2(M)$. Define $V_k = \mathrm{span}\{L(G_k)f\}$ for $k = 0, 1, \ldots$ Then $\{V_k\}_{k=0}^{\infty}$ is a multiresolution of $L^2(M)$ with respect to $\{G_k\}_{k=0}^{\infty}$.*

Proof. The above proposition says that $\{V_k\}_{k=0}^{\infty}$ satisfies MR1. Clearly since $G_k \subseteq G_{k+1}$, we have $V_k \subseteq V_{k+1}$. Thus MR2 is also satisfied. Finally let

$$V = \bigcup_{k=0}^{\infty} V_k = \bigcup_{k=0}^{\infty} \mathrm{span}\{L(G_k)f\} = \mathrm{span}\{L(G)f\}.$$

Thus

$$\overline{V} = \overline{\mathrm{span}\{L(G)f\}} = \overline{\mathrm{span}\{L(M)f\}} = L^2(M)$$

for f is a cyclic vector. $\square$

Remark. A multiresolution given in Theorem 2.2.3 is called stationary.

Now let $\{G_k\}_{k=0}^{\infty}$ be an MR-group sequence of M. Let f be a cyclic vector of $L^2(M)$ and let it generate a stationary multiresolution as in above, namely $V_k = \mathrm{span}\{L(m)f; m \in G_k\}$ for $k = 0, 1, \ldots$ We shall construct an orthonormal wavelet basis of $L^2(M)$ as follows:

For each k, we construct $\phi_k^0 \in V_k$ such that $\{L(G_k)\phi_k^0\}$ is an orthonormal basis for V_k. See Section 2.1. Recall the space W_k, the orthogonal complement of V_k in V_{k+1}. The projection P_k of V_{k+1} onto W_k is given by

$$P_k(q) = q - \sum_{g \in G_k} \langle q, L(g)\phi_k^0 \rangle L(g)\phi_k^0.$$

Note that the map P_k is G_k equivariant, namely $L(m)P_k(q) = P_k(L(m)q)$ for all $m \in G_k$. Since

$$W_k = \mathrm{span}\{P_k(L(m)f); m \in G_{k+1}\} = \mathrm{span}\{P_k(L(m)f); m \in G_{k+1} - G_k\},$$

we see by dimension counting that $\{P_k(L(m)f); m \in G_{k+1} - G_k\}$ is linearly independent, and so form a basis of W_k. Choose a set of representatives $\{x_i : i = 0, 1, \ldots, t_k - 1\}$ for the quotient G_{k+1}/G_k , where $x_0 = e$. Let

$$\psi_k^i = P_k(L(x_i)f) = L(x_i)f - \sum_{g \in G_k} \langle L(x_i)f, L(g)\phi_k^0 \rangle L(g)\phi_k^0, \quad 1 \leqslant i \leqslant t_k - 1.$$

Since $P_k(L(G_k x_i)f) = L(G_k)\psi_k^i$, we see that $\{L(G_k)\psi_k^i\}_{1 \leqslant i \leqslant t_k - 1}$ form a basis of W_k. We construct ϕ_k^i, $1 \leqslant i \leqslant t_k - 1$, such that $\{L(G_k)\phi_k^i\}_{1 \leqslant i \leqslant t_k - 1}$ is an orthonormal basis of W_k (see Section 2.1). Then

$$\{L(G_0)\phi_0^0\} \cup \left\{ \bigcup_{k=0}^{\infty} \bigcup_{l=1}^{t_k - 1} \{L(G_k)\phi_k^l\} \right\}$$

is an orthonormal basis of $L^2(M)$.

3. A non-abelian example

Consider the infinite dihedral group D_∞ and its subgroups D_n defined to be the following subgroups of $GL_2(\mathbb{C})$ with generators:

$$D_\infty = \langle r_\alpha, s \mid \alpha \in [0, 2\pi) \rangle, \quad D_n = \langle r_{2\pi/n}, s \rangle,$$

where

$$r_\alpha = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$$

for $\alpha \in [0, 2\pi)$, and

$$s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

A moment's reflection shows that D_∞ is isomorphic to the semi- direct product $\mathbb{T} \rtimes \mathbb{Z}_2$, where $\mathbb{T} = \{r_\alpha \mid \alpha \in [0, 2\pi)\}$ and $\mathbb{Z}_2 = \{1, s\}$, and $D_n \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2$. In fact, D_∞ is isomorphic to O_2, the full orthogonal group in dimension two. Also as a set, $D_\infty = \{r_\alpha, sr_\alpha \mid \alpha \in [0, 2\pi)\}$ and $D_n = \{r_{2\pi k/n}, sr_{2\pi k/n} \mid k = 0, 1, \ldots, n-1\}$.

Clearly $\{D_{2^k}\}_{k=0}^{\infty}$ is an MR-group sequence of D_∞ (cf. Example b in Section 1).

3.1. *Facts about representations of D_∞ and D_n, n even*

We refer the reader to [6] for the following facts.

(a) *Characters of D_∞*. There are two irreducible representations of D_∞ of degree one with the characters:

$$\eta_1(x) = 1, \ x \in D_\infty, \quad \eta_2(x) = \det(x) = \begin{cases} 1, & x = r_\alpha, \\ -1, & x = sr_\alpha. \end{cases}$$

For each natural number $m \in \mathbb{N}$, there is an irreducible representation π^m of degree two with the character

$$\chi_m(x) = \begin{cases} 2\cos(m\alpha), & x = r_\alpha, \\ 0, & x = sr_\alpha. \end{cases}$$

The representation π^m has the following matrix realization:

$$\pi^m(r_\alpha) = \begin{pmatrix} e^{im\alpha} & 0 \\ 0 & e^{-im\alpha} \end{pmatrix}, \quad \pi^m(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus

$$\pi^m(sr_\alpha) = \begin{pmatrix} 0 & e^{-im\alpha} \\ e^{im\alpha} & 0 \end{pmatrix}.$$

All irreducible representations of D_∞ are of the form $\eta_1, \eta_2, \{\pi^m\}_{m\in\mathbb{N}}$.

(b) *Characters of D_n, n even*. There are four irreducible representations of D_n of degree one with the characters:

$$\gamma_1(x) = 1, \quad x \in D_n, \qquad \gamma_2(x) = \begin{cases} 1, & x = r_{2\pi k/n}, \\ -1, & x = sr_{2\pi k/n}, \end{cases}$$

$$\gamma_3(x) = (-1)^k, \quad x = \begin{cases} r_{2\pi k/n}, \\ sr_{2\pi k/n}, \end{cases} \qquad \gamma_4(x) = \begin{cases} (-1)^k, & x = r_{2\pi k/n}, \\ (-1)^{k+1}, & x = sr_{2\pi k/n}. \end{cases}$$

For each $j = 1, 2, \ldots, (n/2) - 1$, there is an irreducible representation σ^j of degree two with the character

$$\xi_j(x) = \begin{cases} 2\cos(\frac{2\pi jk}{n}), & x = r_{2\pi k/n}, \\ 0, & x = sr_{2\pi k/n}. \end{cases}$$

All irreducible representations of D_n for n even are of the form

$$\{\gamma_i\}_{i=1}^4, \{\sigma^j\}_{j=1}^{n/2-1}.$$

We now describe how representations of D_∞ behave under restriction to D_n for n even. We have

$$\eta_1|D_n = \gamma_1, \ \eta_2|D_n = \gamma_2,$$

For any $m \in \mathbb{N}$, let $0 \leqslant j \leqslant n/2$ be such that $m \equiv \pm j \pmod{n}$. Then

$$\chi_m|_{D_n} = \begin{cases} \xi_j, & j = 1, \ldots, n/2 - 1, \\ \gamma_1 + \gamma_2, & j = 0, \\ \gamma_3 + \gamma_4, & j = n/2. \end{cases}$$

3.2. *A multiresolution of $L^2(D_\infty)$ with respect to $\{D_{2^k}\}_{k=0}^\infty$*

Recall the irreducible representation π^m of D_∞ for $m \in \mathbb{N}$. We define the matrix coefficients $\{\pi_{ij}^m\}_{i,j=1}^2$ of π^m by

$$\pi^m(x) = \begin{pmatrix} \pi_{11}^m(x) & \pi_{12}^m(x) \\ \pi_{21}^m(x) & \pi_{22}^m(x) \end{pmatrix}$$

for $x \in D_\infty$. Thus we have

$$\pi_{11}^m(x) = \begin{cases} \mathrm{e}^{\mathrm{i}m\alpha}, & x = r_\alpha, \\ 0, & x = sr_\alpha, \end{cases} \quad \pi_{12}^m(x) = \begin{cases} 0, & x = r_\alpha, \\ \mathrm{e}^{-\mathrm{i}m\alpha}, & x = sr_\alpha. \end{cases}$$

$$\pi_{21}^m(x) = \begin{cases} 0, & x = r_\alpha, \\ \mathrm{e}^{\mathrm{i}m\alpha}, & x = sr_\alpha, \end{cases} \quad \pi_{22}^m(x) = \begin{cases} \mathrm{e}^{-\mathrm{i}m\alpha}, & x = r_\alpha, \\ 0, & x = sr_\alpha. \end{cases}$$

For even n, define the functions $\eta_3^n, \eta_4^n, \zeta_3^n, \zeta_4^n \in L^2(D_\infty)$ by

$$\eta_3^n(x) = \left(\pi_{11}^{n/2} + \pi_{21}^{n/2}\right)(x) = \mathrm{e}^{\mathrm{i}(n/2)\alpha} \quad \text{if } x = r_\alpha, sr_\alpha,$$

$$\eta_4^n(x) = \left(\pi_{11}^{n/2} - \pi_{21}^{n/2}\right)(x) = \begin{cases} \mathrm{e}^{\mathrm{i}(n/2)\alpha}, & x = r_\alpha, \\ -\mathrm{e}^{\mathrm{i}(n/2)\alpha}, & x = sr_\alpha. \end{cases}$$

$$\zeta_3^n(x) = \left(\pi_{22}^{n/2} + \pi_{12}^{n/2}\right)(x) = \mathrm{e}^{-\mathrm{i}(n/2)\alpha} \quad \text{if } x = r_\alpha, sr_\alpha,$$

$$\zeta_4^n(x) = \left(\pi_{22}^{n/2} - \pi_{12}^{n/2}\right)(x) = \begin{cases} \mathrm{e}^{-\mathrm{i}(n/2)\alpha}, & x = r_\alpha, \\ -\mathrm{e}^{-\mathrm{i}(n/2)\alpha}, & x = sr_\alpha. \end{cases}$$

Obviously we have

$$L(x)(\eta_1) = \gamma_1(x)\eta_1, \quad L(x)(\eta_2) = \gamma_2(x)\eta_2, \qquad x \in D_n.$$

Here and as before L stands for the left regular representation of D_∞ on $L^2(D_\infty)$. A direct computation (on generators of D_n) also yields

$$L(x)(\eta_3^n) = \gamma_3(x)\eta_3^n, \quad L(x)(\zeta_3^n) = \gamma_3(x)\zeta_3^n, \qquad x \in D_n,$$

$$L(x)(\eta_4^n) = \gamma_4(x)\eta_4^n, \quad L(x)(\zeta_4^n) = \gamma_4(x)\zeta_4^n, \qquad x \in D_n.$$

For $m \in \mathbb{N}$, denote by $L(m)$ the isotypic component of π^m in $L^2(D_\infty)$. By Peter–Weyl Theory, $\dim L(m) = 4$ and it is spanned by $\{\pi_{ij}^m\}_{i,j=1}^2$.

Theorem 3.2.1. *Define*

$$V_0 = \mathbb{C}\eta_1 \oplus \mathbb{C}\eta_2,$$
$$V_k = \mathbb{C}\eta_1 \oplus \mathbb{C}\eta_2 \oplus \mathbb{C}\eta_3^{2^k} \oplus \mathbb{C}\eta_4^{2^k} \oplus \left(\bigoplus_{m=1}^{2^{k-1}-1} L(m)\right), \quad k = 1, 2, \ldots$$

Then $\{V_k\}_{k=0}^\infty$ is a multiresolution of $L^2(D_\infty)$ with respect to $\{D_{2^k}\}_{k=0}^\infty$.

Proof. MR1: Observe that V_k contains exactly each irreducible representation σ of D_{2^k} with multiplicity equal to $\dim \sigma$. Thus $V_k \cong L^2(D_{2^k})$ as D_{2^k} modules. Take $\phi_0 = \eta_1 + \eta_2$, and $\phi_k = \eta_1 + \eta_2 + \eta_3^{2^k} + \eta_4^{2^k} + \sum_{m=1}^{2^{k-1}-1} \chi_m$ for $k \geqslant 1$, then $V_k = \mathrm{span}\{L(D_{2^k})\phi_k\}$ by the characterization of cyclic vectors on a finite group. See Theorem 1.1.6.

MR2: We have by definition

$$V_k = \mathbb{C}\eta_1 \oplus \mathbb{C}\eta_2 \oplus \mathbb{C}\eta_3^{2^k} \oplus \mathbb{C}\eta_4^{2^k} \oplus \left(\bigoplus_{m=1}^{2^{k-1}-1} L(m)\right),$$

$$V_{k+1} = \mathbb{C}\eta_1 \oplus \mathbb{C}\eta_2 \oplus \mathbb{C}\eta_3^{2^{k+1}} \oplus \mathbb{C}\eta_4^{2^{k+1}} \oplus \left(\bigoplus_{m=1}^{2^k-1} L(m)\right).$$

Observe that

$$\begin{aligned}L(2^{k-1}) &= \mathrm{span}\{\pi_{11}^{2^{k-1}}, \pi_{12}^{2^{k-1}}, \pi_{21}^{2^{k-1}}, \pi_{22}^{2^{k-1}}\}\\ &= \mathrm{span}\{\eta_3^{2^k}, \eta_4^{2^k}, \zeta_3^{2^k}, \zeta_4^{2^k}\}.\end{aligned}$$

Therefore we have $V_k \subset V_{k+1}$.

MR3: Clearly $\cup_{k=0}^{\infty} V_k$ contains D_∞-isotypic components of η_1, η_2, namely $\mathbb{C}_{\eta_1}, \mathbb{C}_{\eta_2}$, and $L(m)$, the π^m isotypic component of $L^2(D_\infty)$ for any $m \in \mathbb{N}$. Thus

$$\overline{\bigcup_{k=0}^{\infty} V_k} = L^2(D_\infty). \qquad \square$$

To construct a wavelet basis of $L^2(D_\infty)$, we write $V_{k+1} = V_k \oplus W_k$, where W_k is the D_{2^k}-invariant subspace defined by

$$W_0 = \mathbb{C}\eta_3^2 \oplus \mathbb{C}\eta_4^2,$$

$$W_k = \mathbb{C}\eta_3^{2^{k+1}} \oplus \mathbb{C}\eta_4^{2^{k+1}} \oplus \mathbb{C}\zeta_3^{2^k} \oplus \mathbb{C}\zeta_3^{2^k} \oplus \left(\bigoplus_{m=2^{k-1}+1}^{2^k-1} L(m)\right), \quad k \geqslant 1.$$

We then have $L^2(D_\infty) = V_0 \oplus (\oplus_{k=0}^{\infty} W_k)$, and $W_k \cong L^2(D_{2^k})$ as D_{2^k} modules (cf. Section 2.1). Notice that

$$\eta_3^{2^{k+1}}\Big|D_{2^k} = \gamma_1, \qquad \eta_4^{2^{k+1}}\Big|D_{2^k} = \gamma_2,$$

and $\zeta_3^{2^k}, \zeta_4^{2^k}$ transform according to the characters γ_3, γ_4 of D_{2^k}. Notice also that

$$\bigoplus_{m=2^k-1+1}^{2^k-1} L(m) = \bigoplus_{j=1}^{2^{k-1}-1} L(2^k - j) \cong \bigoplus_{j=1}^{2^{k-1}-1} L(j)$$

as D_{2^k} modules.

Let

$$\phi_0 = \eta_1 + \eta_2,$$
$$\psi_0 = \eta_3^2 + \eta_4^2,$$
$$\psi_k = \eta_3^{2^{k+1}} + \eta_3^{2^{k+1}} + \zeta_3^{2^k} + \zeta_4^{2^k} + \sum_{m=2^{k-1}+1}^{2^k-1} \chi_m, \quad k \geqslant 1.$$

Then $V_0 = \mathrm{span}\{L(D_0)\phi_0\}$ and $W_k = \mathrm{span}\{L(D_{2^k})\psi_k\}$ for all $k \geqslant 0$, and so

$$\left\{L(D_0)\phi_0\} \cup \left\{\bigcup_{k=0}^{\infty}\{L(D_{2^k})\psi_k\right\}\right\}$$

is a wavelet basis of $L^2(D_\infty)$ with respect to the MR-group sequence $\{D_{2^k}\}_{k=0}^{\infty}$.

4. Concluding remarks

(a) By using the classification of finite subgroups of SO_3, one can show that SO_3 does not possess any MR-group sequence. We outline such an argument. As a finite subgroup of SO_3, each G_k must be isomorphic to one of the following subgroups: (1) $\mathbb{Z}_n$, (2) D_n, (3) A_4, (4) A_5, (5) S_4 (see [1], p.18). The dimensions of irreducible representations of all the groups listed above have a uniform bound. If $\{G_k\}_{k=0}^{\infty}$ is an MR-group sequence of SO_3, then any irreducible representation of SO_3 is irreducible when restricted to G_k for k large enough. Thus the dimensions of irreducible representations of G_k for all k will have to be unbounded, which is a contradiction.

(b) In view of (a), our theory of multiresolution on compact groups is probably more suited for compact groups which are semi-direct products than for compact semi-simple groups such as SO_3. Our example in §3 illustrates this point and as well as the usefulness of the notion of multiresolution for the infinite dihedral group, which is the semi-direct product of the torus with $\mathbb{Z}_2$.

(c) Finally we would like to point out that most of our results actually do not depend on the hypothesis that the MR-group sequence $\{G_k\}_{k=0}^{\infty}$ exists in M. More specifically, with the exception of Section 1.3. Characterization of MR3, our results are stated in the general setup of either a compact group K, or a compact group M and a finite subgroup H. See for example Theorems 1.1.6, 2.1.3, and 2.2.2. They are thus of independent interest.

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References

[1] D. Gurarie, Symmetries and laplacians, Mathematics Studies, No. 174, North-Holland, Amsterdam, 1992.
[2] Y.W. Koh, S.L. Lee, H.H. Tan, Periodic orthogonal splines and wavelets, Appl. Comput. Harmonic Anal. 2 (1995) 201–218.
[3] A. Lim, Wavelets on compact groups, master's thesis, National University of Singapore, 1994.

[4] S. Mallat, Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbb{R})$, Trans. Amer. Math. Soc. 315 (1989) 69–87.
[5] W. Rudin, Functional Analysis, McGraw-Hill, New York, 1973.
[6] J.P. Serre, Linear Representations of Finite Groups, Springer, Berlin, 1977.