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On product of companion matrices

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ABSTRACT

This paper describes an explicit combinatorial formula for the product of companion matrices. The method relies on the connections between matrix algebra and associated combinatorial structures to enumerate the paths in an unweighted digraph. As an application, we obtain bases for the solution space of the linear difference equation with variable coefficients.

1. Introduction

The product of companion matrices naturally arises in the solution of linear difference equation, in the studies of certain random walks and Markov Chains. There are also interests in the eigenvalues of the product of companion matrices (see [6,7]). In this paper, we use a combinatorial method to find an explicit formula for the entries of the product of companion matrices. The formula is then applied to solve the general linear difference equation with variable coefficients.

Taking powers of a matrix using combinatorics and weighted digraphs are discussed in Section 3.1 of [1] and further studied in [3] for the companion matrix case. Unaware of these results, the authors discovered a combinatorial method of taking product of companion matrices (of the same size) from

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a new perspective. The different perspective yields the same digraph in [3] but without weighted edges. This allows the combinatorial method to extend beyond the homogeneous power of a companion matrix to the (non-homogeneous) product of companion matrices. The authors' construction also highlights natural connections between matrix algebra and the paths of the associated digraph. Moreover the vertices of the digraph correspond to the summands in the decomposition of a companion matrix over its rows.

As one may expect, the answer for an explicit formula for the entries of the product of companion matrices should be quite complicated. However, the use of a digraph has greatly helped in managing and organizing our computations. Moreover, the interesting involvement of the permutation group in the final answer allows us to write a reasonably short formula for the product of companion matrices and a simple description of the solution space of a linear difference equation. Indeed, our method is another ode to the usefulness of graphs and groups.

We should point out that although explicit solution for general linear difference equations are given in [4], they appear to be unmotivated and no methods of solution are discussed. In contrast, the method of solution in this paper connects the solutions of linear difference equation with variable coefficients to enumerative combinatorics in an associated digraph. In addition, we also show that the solution space of a linear homogeneous difference equation is given by the linear combinations of the integer translates of a single function $T_{k,m}(\vec{c}, r)$ (see Section 2, Corollaries 6.1 and 6.2).

This article is organized as follows. Section 2 defines notations and states the key formula for the product of companion matrices. Section 3 explains the construction of the digraph associated to companion matrices of the same size. Section 4 gives important reduction relations and describes the algebraic structures that simplify our computations (see Lemma 4.1 and Proposition 4.2). In Section 5 we put together all results to compute the product of companion matrices. We apply the product formula for the companion matrices to solve the linear difference equation in Section 6. We make some concluding remarks in Section 7.

2. Notations and key formula

In this section, we define notations and state the formula for the product of companion matrices. We will adopt all notations established in this section throughout the paper. Let $\mathbb{N}_0$ denote the set of non-negative integers. For each positive integers k, m and $1 \leqslant j \leqslant k$, define the set:

$$Q(k, m) = \{(a_1, a_2, \dots, a_k) \in {\mathbb{N}_0}^k; a_1 + 2a_2 + \cdots + ja_j + \cdots + ka_k = m\}.$$

For each $\vec{a} = (a_1, a_2, \dots, a_k) \in Q(k, m)$, define the integer-valued function $\chi_{\vec{a}}$ on the set of integers $\{1, 2, \dots, a_1 + a_2 + \cdots + a_k\}$ by

$$\chi_{\vec{a}}(s) = \begin{cases} 1 & 1 \leqslant s \leqslant a_1 \\ 2 & a_1 + 1 \leqslant s \leqslant a_1 + a_2 \\ \vdots & \\ j & a_1 + \cdots + a_{j-1} + 1 \leqslant s \leqslant a_1 + \cdots a_{j-1} + a_j \\ \vdots & \\ k & a_1 + \cdots + a_{k-1} + 1 \leqslant s \leqslant a_1 + \cdots a_{k-1} + a_k. \end{cases}$$

We note here that for any $1 \leqslant j \leqslant k$, if $a_j = 0$ then j will not be in the range of $\chi_{\vec{a}}$. Moreover, we have:

$$\sum_{s=1}^{a_1+a_2+\cdots+a_k} \chi_{\vec{a}}(s) = a_1(1) + a_2(2) + \cdots + a_k(k) = m.$$

The purpose of defining $\chi_{\vec{a}}$ is to aid in listing all permutations of a given multiset (a generalization of set where members are allowed to repeat). For example, $\chi_{\vec{a}}$ is associated to the multiset

$$R = \{1, 1, \ldots, 1, 2, 2, \ldots, 2, \ldots, k, k, \ldots, k\}$$

with a_1 copies of 1's, a_2 copies of 2's, ..., and a_k copies of k's. By definition of $\chi_{\vec{a}}$, we can rewrite the set R:

$$\begin{aligned} R = \{&\chi_{\vec{a}}(1), \chi_{\vec{a}}(2), \ldots, \chi_{\vec{a}}(a_1), \chi_{\vec{a}}(a_1+1), \chi_{\vec{a}}(a_1+2), \ldots, \chi_{\vec{a}}(a_1+a_2), \ldots, \\ &\chi_{\vec{a}}(a_1+\cdots+a_{k-1}), \chi_{\vec{a}}(a_1+\cdots+a_{k-1}+1), \ldots, \chi_{\vec{a}}(a_1+\cdots+a_k)\} \\ = \{&\chi_{\vec{a}}(j); 1 \leqslant j \leqslant (a_1+\cdots+a_k)\}. \end{aligned}$$

Let S_n denote the symmetric group of order n. For $\sigma \in S_n$, let $\sigma(s)$ denote the image of s under σ. To list all permutations (with repeats) of the multiset R, we consider the sets, for $\sigma \in S_{a_1+\cdots+a_k}$,

$$R_\sigma = \{\chi_{\vec{a}}(\sigma(j)); 1 \leqslant j \leqslant (a_1+\cdots+a_k)\}.$$

The total number of **distinct** R_σ is given by

$$\frac{|S_{a_1+a_2+\cdots+a_k}|}{a_1!a_2!\cdots a_k!} = \binom{a_1+a_2+\cdots+a_k}{a_1, a_2, \ldots, a_k}.$$

This systematic way of listing permutations of a given multiset will help in keeping our notations simple in some rather complicated formulas for products of companion matrices (below) and solutions for linear difference equation (see Section 6).

We state our formula for products of companion matrices here. Let $c_1(m), c_2(m), \ldots, c_k(m)$ be real-valued functions over the integers. Set $\vec{c}(m) = (c_1(m), c_2(m), \ldots, c_k(m))$. For $m \in \mathbb{N}$, define the function $T_{k,m}(\vec{c}, r)$:

$$T_{k,m}(\vec{c}, r) = \sum_{\vec{a} \in Q(k,m)} \frac{1}{a_1!a_2!\cdots a_k!} \sum_{\sigma \in S_{a_1+\cdots+a_k}} \prod_{n=1}^{a_1+\cdots+a_k} c_{\chi_{\vec{a}}(\sigma(n))}\left(r + \sum_{s=1}^{n-1} \chi_{\vec{a}}(\sigma(s))\right).$$

Since $\vec{c}$ is fixed throughout this paper, we shall write $T_{k,m}(\vec{c}, r)$ as $T_{k,m}(r)$. The role of r will become clear in Section 5. Moreover, since $T_{k,m}(0)$ repeatedly occurs in the solution for a linear difference equation, we further simplify notations and write $T_{k,m}(0)$ as $T_{k,m}$. Define the companion matrix $C(m)$ associated to the vector-valued function $\vec{c}(m) = (c_1(m), c_2(m), \ldots, c_k(m))$ by

$$C(m) = \begin{bmatrix} c_1(m) & c_2(m) & \ldots & c_{k-1}(m) & c_k(m) \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}.$$

Theorem 2.1. *Let m_{ij} be the (i,j)-entry of the product $C(0)C(1)\cdots C(m)$. Then $m_{ij} = \sum_{p=j}^{k} c_p(m-i+1+j-p)T_{k,m-i+1+j-p}(i-1)$.*

Theorem 2.1 will be proved in Section 5. It will be used in Section 6 to solve linear difference equations.

3. The companion digraph

Recall the sequence of companion matrices $C(n)$ associated to the vector-valued function $\vec{c}(n) = (c_1(n), c_2(n), \ldots, c_k(n))$. The first step in finding a formula for the matrix product $C(1)C(2)\cdots C(m)$ is to associate an unweighted digraph G to the companion matrix $C(n)$. This digraph with weights also appeared in [3] to take powers of a single companion matrix. However, we obtain G from a different perspective using matrix algebra. We call this digraph the companion digraph of $C(n)$.

Let $E_{i,j}$ be the matrix with all entries 0 except 1 at the (i,j)-entry. Define the matrices:

$$U_j = E_{j,j-1}; \quad \text{and} \quad U_1(n) = \begin{bmatrix} c_1(n) & c_2(n) & \dots & c_k(n) \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} = \sum_{p=1}^{k} c_p(n) E_{1,p}.$$

Then we have $C(n) = (U_1(n) + U_2 + U_3 + \cdots + U_k)$. We will construct a graph G to keep track of vanishing and non-vanishing products between the matrices $U_1(n)$; $(1 \leqslant n \leqslant m)$, U_2, ..., U_k. We now compute the pairwise multiplication of these matrices below.

$$\begin{aligned} U_1(s)U_1(t) &= \left(\sum_{p=1}^{k} c_p(s)E_{1,p}\right) \cdot \left(\sum_{p=1}^{k} c_p(t)E_{1,p}\right) = c_1(s)E_{1,1} \cdot \left(\sum_{p=1}^{k} c_p(t)E_{1,p}\right) \\ &= c_1(s) \cdot \left(\sum_{p=1}^{k} c_p(t)E_{1,p}\right) = c_1(s)U_1(t), \\ U_2 \cdot U_1(t) &= E_{2,1} \cdot \left(\sum_{p=1}^{k} c_p(t)E_{1,p}\right) = \sum_{p=1}^{k} c_p(t)E_{2,p}. \end{aligned}$$

For each fixed $3 \leqslant i \leqslant k$ and any $2 \leqslant j \leqslant k$,

$$U_i \cdot U_1(t) = E_{i,i-1} \cdot \left(\sum_{p=1}^{k} c_p(t)E_{1,p}\right) = \sum_{p=1}^{k} c_p(t)E_{i,i-1}E_{1,p} = 0,$$

$$U_1(t) \cdot U_j = \left(\sum_{p=1}^{k} c_p(t)E_{1,p}\right) \cdot E_{j,j-1} = c_j(t)E_{1,j-1},$$

$$U_2 \cdot U_j = E_{2,1} \cdot E_{j,j-1} = 0,$$

$$U_i \cdot U_j = E_{i,i-1} \cdot E_{j,j-1} = \begin{cases} U_i \cdot U_{i-1} = E_{i,i-2} \neq 0; & j = i-1. \\ 0; & j \neq i-1. \end{cases}$$

We summarize our observations. For each $1 \leqslant i, j \leqslant k$ and any integers s and t:

$$U_i \cdot U_j = \begin{cases} U_1(s)U_1(t) = c_1(s)U_1(t); & \\ U_1(t) \cdot U_j = c_j(t)E_{1,j-1}; & 2 \leqslant j \leqslant k. \\ U_2 \cdot U_1(t) = \sum_{p=1}^{k} c_p(t)E_{2,p}; & \\ E_{i,i-2}; & 3 \leqslant i \leqslant k \text{ and } j = i-1. \\ 0; & \text{otherwise.} \end{cases} \tag{1}$$

Without loss of generality, we shall assume that $c_1, c_2, \ldots, c_k$ are all non-trivial functions. Then $U_1(s) \cdot U_1(t)$ and $U_1(t) \cdot U_j$ are non-zero for any j, s and t. Moreover the vanishing (and non-vanishing) relations

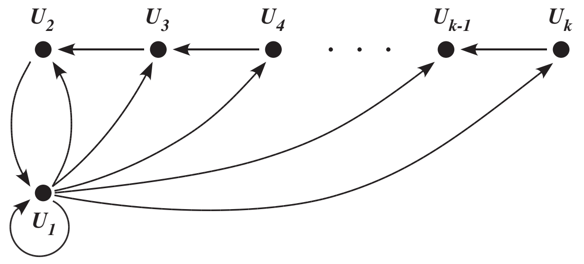

Fig. 1. Companion digraph for matrix $C(n)$.

above are dependent only on the subscripts i and j but independent of s and t. We now construct the graph G associated to all companion matrices $C(n)$ to keep track of non-vanishing products of $U_1(n)$; $(1 \leqslant n \leqslant m)$, U_2, ..., U_k. Since vanishing property is independent of n, we shall write $U_1(n)$ as U_1. Consider the digraph G with k vertices each corresponding to matrices U_i $(1 \leqslant i \leqslant k)$. Without confusion, we write the vertex set

$$V(G) = \{U_i; 1 \leqslant i \leqslant k\}.$$

Let $A, B \in V(G)$. The **directed** edge AB is in G if and only if the corresponding matrix multiplication $AB \neq 0$. Therefore according to the pairwise multiplication of U_i $(1 \leqslant i \leqslant k)$ above, we see that the edge set is

$$E(G) = \{U_iU_{i-1}, U_1U_i, U_1U_1; 2 \leqslant i \leqslant k\}.$$

Fig. 1 depicts the graph G. We call G the companion digraph for matrix C for a $k \times k$ companion matrix. We note here that if any of the functions $c_1, c_2, \ldots, c_k$ is identically zero then the graph G will reduce in size. But in this paper we will consider the most general case where $c_1, c_2, \ldots, c_k$ are all non-trivial functions.

We roughly indicate here how the digraph G would be helpful. Consider the product $C(1)C(2)\cdots C(m)$. We compute

$$\begin{aligned} C(1)C(2)\cdots C(m) &= \textstyle\prod_{n=1}^{m}(U_1(n) + U_2 + U_3 + \cdots + U_k) \\ &= \textstyle\sum A_1A_2\cdots A_m, \end{aligned}$$

here $A_n \in \{U_1(n), U_2, U_3, \cdots, U_k\}$ for $1 \leqslant n \leqslant m$ and the sum is over all products $A_1A_2\cdots A_m$. By the relations in (1), the product $A_1A_2\cdots A_m$ is non-zero only if it admits a path of length $(m-1)$ in the digraph G. Thus to simplify the sum, we use the digraph and the relations in (1) to pick up all non-zero terms. This computation could be systematically done and its crux is the useful order-reducing relations given in the next section.

4. Some useful products and order-reducing relations

In this section, we prove order-reducing relations useful for computing a formula for the matrix product $C(1)C(2)\cdots C(m)$.

For $2 \leqslant j \leqslant k$, define the product $\overline{U_j} = U_jU_{j-1}U_{j-2}\cdots U_2$. Note that $\overline{U_2} = U_2$. We compute: $\overline{U_j} = E_{j,j-1}E_{j-1,j-2}\cdots E_{2,1} = E_{j,1}$. Moreover, we have:

$$\overline{U_j}U_1(n) = E_{j,1}\sum_{i=1}^{k} c_i(n)E_{1,i} = \sum_{i=1}^{k} c_i(n)E_{j,i}.$$

From Eq. (1), we make the following critical observations. For easy reference later, we state them here as a lemma:

Lemma 4.1. *For $s, t \in \mathbb{N}_0$ and $2 \leqslant i, j \leqslant k$ we have:*

(1) $U_1(s) \cdot \overline{U_i}U_1(t) = c_i(s)U_1(t)$,
(2) $U_1(s) \cdot U_1(t) = c_1(s)U_1(t)$,
(3) $\overline{U_j}U_1(s) \cdot \overline{U_i}U_1(t) = c_i(s)\overline{U_j}U_1(t)$,
(4) $\overline{U_j}U_1(s) \cdot U_1(t) = c_1(s)\overline{U_j}U_1(t)$.

We observe from Lemma 4.1 that the set of matrices

$$M = \{\overline{U_i}U_1(n), U_1(n) : n \in \mathbb{N} \text{ and } 2 \leqslant i \leqslant k\}$$

forms an algebra over the ring of polynomial generated by $c_1, c_2, \cdots c_k$. The algebraic structures in this observation captures the key properties that make it possible to compute formulas in Theorem 4.4. We state here the interesting observation as a proposition:

Proposition 4.2. *Let R be the ring of polynomial generated by $c_1, c_2, \cdots c_k$ over the complex numbers. Define M as above and set $N_1 = \{U_1(n) : n \in \mathbb{N}\}$, and $N_i = \{\overline{U_i}U_1(n) : n \in \mathbb{N}\}$ for $2 \leqslant i \leqslant k$. Consider the R-module $R[M]$ generated by the set M. Then $R[M]$ is an R-algebra. Moreover, $R[N_i]$ is a right ideal of $R[M]$ for $1 \leqslant i \leqslant k$.*

Recall that at the end of Section 3 we see that to evaluate the matrix product $C(1)C(2)\cdots C(m)$, it amounts to finding non-zero products admitted by paths of length $(m-1)$ in G. We shall do this computation here.

For $\vec{a} = (a_1, a_2, \cdots, a_k) \in Q(k, m)$, consider the non-zero term $P_1(\vec{a})$ admitted by the following path $L_1(\vec{a})$ in the digraph G ending at U_1:

$$L_1(\vec{a}) = (U_1)^{a_1}(\overline{U_2}U_1)^{a_2}\cdots(\overline{U_r}U_1)^{a_r}\cdots(\overline{U_k}U_1)^{a_k}. \tag{2}$$

Note that the above path is a closed path (starting at U_1) only when $a_1 \geqslant 1$. In any case, as we transverse through the above path, we are transversing through closed paths based at U_1 of increasing length. Moreover the length of $L_1(\vec{a})$ is $a_1 + 2a_2 + \cdots + ka_k = m - 1$. We give an example for $L_1(\vec{a})$ below and illustrate how to write its associated $P_1(\vec{a})$.

Example 4.3. Consider the companion matrix $C(n)$ associated to the 4-tuple function $\vec{c}(n) = (c_1(n), c_2(n), c_3(n), c_4(n))$. Consider also $C(1)C(2)\cdots C(13)$. Then in the notations above $m = 13$. Fix $\vec{a} = (2, 2, 1, 1) \in Q(4, 13)$. Then $\chi_{\vec{a}}$ is given by

$$\chi_{\vec{a}}(s) = \begin{cases} 1 & 1 \leqslant s \leqslant 2 \\ 2 & 3 \leqslant s \leqslant 4 \\ 3 & s = 5 \\ 4 & s = 6 \end{cases}.$$

The companion graph for $C(n)$ is given in Fig. 2. The multiset associated to $\chi_{\vec{a}}$ is given by $R = \{1, 1, 2, 2, 3, 4\}$. Then the path $L_1(\vec{a})$ is

$$(U_1)^2(U_2U_1)^2(U_3U_2U_1)(U_4U_3U_2U_1).$$

Then the associated matrix product $P_1(\vec{a})$ is given by

$$U_1(1)\cdot U_1(2)\cdot(U_2U_1(4))\cdot(U_2U_1(6))\cdot(U_3U_2U_1(9))\cdot(U_4U_3U_2U_1(13)).$$

Using Lemma 4.1, we evaluate:

$$\begin{aligned} P_1(\vec{a}) &= U_1(1)\cdot U_1(2)\cdot(\overline{U_2}U_1(4))\cdot(\overline{U_2}U_1(6))\cdot(\overline{U_3}U_1(9))\cdot(\overline{U_4}U_1(13)) \\ &= c_1(1)c_2(2)c_2(4)c_3(6)c_4(9)\cdot U_1(13). \end{aligned} \tag{3}$$

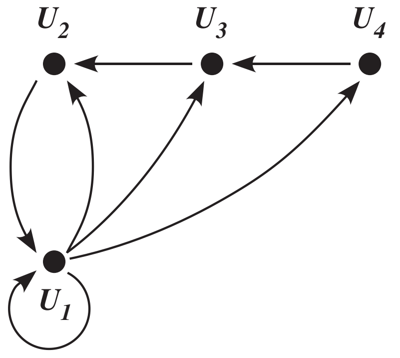

Fig. 2. Companion digraph for matrix $C(n)$.

Using $\chi_{\vec{a}}$, we could write

$$P_1(\vec{a}) = U_{\chi_{\vec{a}}(1)}(1) \cdot U_{\chi_{\vec{a}}(2)}(2) \cdot (\overline{U_{\chi_{\vec{a}}(3)}}U_1(4))$$
$$(\overline{U_{\chi_{\vec{a}}(4)}}U_1(6)) \cdot (\overline{U_{\chi_{\vec{a}}(5)}}U_1(9)) \cdot (\overline{U_{\chi_{\vec{a}}(6)}}U_1(13)).$$

Further applying $\chi_{\vec{a}}$, we have:

$$\begin{aligned} P_1(\vec{a}) &= U_{\chi_{\vec{a}}(1)}(\chi_{\vec{a}}(1)) \cdot U_{\chi_{\vec{a}}(2)}\left(\sum_{s=1}^{2}\chi_{\vec{a}}(s)\right) \cdot \overline{U_{\chi_{\vec{a}}(3)}}U_1\left(\sum_{s=1}^{3}\chi_{\vec{a}}(s)\right) \\ &\quad \times \overline{U_{\chi_{\vec{a}}(4)}}U_1\left(\sum_{s=1}^{4}\chi_{\vec{a}}(s)\right) \cdot \overline{U_{\chi_{\vec{a}}(5)}}U_1\left(\sum_{s=1}^{5}\chi_{\vec{a}}(s)\right) \cdot \overline{U_{\chi_{\vec{a}}(6)}}U_1\left(\sum_{s=1}^{6}\chi_{\vec{a}}(s)\right) \\ &= \prod_{n=1}^{6}\overline{U_{\chi_{\vec{a}}(n)}}U_1\left(\sum_{s=1}^{n}\chi_{\vec{a}}(s)\right). \end{aligned}$$

Here note that we adopt the convention $\overline{U_1}U_1(n) = U_1(n)$. Applying Lemma 4.1, we have:

$$\begin{aligned} P_1(\vec{a}) &= c_{\chi_{\vec{a}}(2)}(\chi_{\vec{a}}(1))c_{\chi_{\vec{a}}(3)}\left(\sum_{s=1}^{2}\chi_{\vec{a}}(s)\right)c_{\chi_{\vec{a}}(4)}\left(\sum_{s=1}^{3}\chi_{\vec{a}}(s)\right) \\ &\quad \times c_{\chi_{\vec{a}}(5)}\left(\sum_{s=1}^{4}\chi_{\vec{a}}(s)\right)c_{\chi_{\vec{a}}(6)}\left(\sum_{s=1}^{5}\chi_{\vec{a}}(s)\right) \cdot U_1\left(\sum_{s=1}^{6}\chi_{\vec{a}}(s)\right) \\ &= \prod_{n=2}^{6}c_{\chi_{\vec{a}}(n)}\left(\sum_{s=1}^{n-1}\chi_{\vec{a}}(s)\right) \cdot \overline{U_{\chi_{\vec{a}}(1)}}U_1(13) \end{aligned}$$

where $\overline{U_{\chi_{\vec{a}}(1)}}U_1(13) = \overline{U_1}U_1(13) = U_1(13)$.

In general, the non-zero term $P_1(\vec{a})$ admitted by the following path $L_1(\vec{a})$ in the digraph G ending at U_1:

$$L_1(\vec{a}) = (U_1)^{a_1}(\overline{U_2}U_1)^{a_2}\cdots(\overline{U_r}U_1)^{a_r}\cdots(\overline{U_k}U_1)^{a_k} \tag{4}$$

is given by:

$$\begin{aligned} P_1(\vec{a}) = \quad & \prod_{n_1=1}^{a_1} U_1\left(\sum_{s=1}^{n_1} \chi_{\vec{a}}(s)\right) \cdot \prod_{n_2=1}^{a_2} \overline{U_2}U_1\left(\sum_{s=1}^{a_1+n_2} \chi_{\vec{a}}(s)\right) \\ & \cdots \prod_{n_r=1}^{a_r} \overline{U_r}U_1\left(\sum_{s=1}^{a_1+\cdots+a_{r-1}+n_r} \chi_{\vec{a}}(s)\right) \\ & \cdots \prod_{n_k=1}^{a_k} \overline{U_k}U_1\left(\sum_{s=1}^{a_1+\cdots+a_{k-1}+n_k} \chi_{\vec{a}}(s)\right). \end{aligned} \tag{5}$$

We could also write:

$$\begin{aligned} P_1(\vec{a}) \quad & = \prod_{m_1=1}^{a_1} U_{\chi_{\vec{a}}(m_1)}\left(\sum_{s=1}^{m_1} \chi_{\vec{a}}(s)\right) \cdot \prod_{m_2=a_1+1}^{a_1+a_2} \overline{U_{\chi_{\vec{a}}(m_2)}}U_1\left(\sum_{s=1}^{m_2} \chi_{\vec{a}}(s)\right) \\ & \cdots \prod_{m_r=a_1+\cdots+a_{r-1}+1}^{a_1+\cdots+a_{r-1}+a_r} \overline{U_{\chi_{\vec{a}}(m_r)}}U_1\left(\sum_{s=1}^{m_r} \chi_{\vec{a}}(s)\right) \\ & \cdots \prod_{m_k=a_1+\cdots+a_{k-1}+1}^{a_1+\cdots+a_{k-1}+a_k} \overline{U_{\chi_{\vec{a}}(m_k)}}U_1\left(\sum_{s=1}^{m_k} \chi_{\vec{a}}(s)\right) \\ & = \prod_{n=1}^{a_1+\cdots+a_k} \overline{U_{\chi_{\vec{a}}(n)}}U_1\left(\sum_{s=1}^{n} \chi_{\vec{a}}(s)\right). \end{aligned} \tag{6}$$

We note here that $\chi_{\vec{a}}(m_r) = r$ for all $1 \leqslant r \leqslant k$. By Lemma 4.1, we could further simplify and write:

$$P_1(\vec{a}) = \prod_{n=2}^{a_1+\cdots+a_k} c_{\chi_{\vec{a}}(n)}\left(\sum_{s=1}^{n-1} \chi_{\vec{a}}(s)\right) \cdot \overline{U_{\chi_{\vec{a}}(1)}}U_1(m). \tag{7}$$

By the same labeling idea using $\chi_{\vec{a}}$ above, we may write $L_1(\vec{a}) = \prod_{n=1}^{a_1+\cdots+a_k} \overline{U_{\chi_{\vec{a}}(n)}}U_1$. To find all other paths of length $(m-1)$ ending at U_1, consider the family of paths in G:

$$L_\sigma(\vec{a}) = \prod_{n=1}^{a_1+\cdots+a_k} \overline{U_{\chi_{\vec{a}}(\sigma(n))}}U_1, \tag{8}$$

where $\vec{a} \in Q(k, m)$ and $\sigma \in S_{a_1+\cdots+a_k}$. This exhaustively lists with repeats all paths of length $(m-1)$ ending at U_1 containing a_1 (U_1)-word, a_2 $(\overline{U_2}U_1)$-word, ..., a_r $(\overline{U_r}U_1)$-word, ..., a_k $(\overline{U_k}U_1)$-word. The number of such path is

$$\frac{|S_{a_1+a_2+\cdots+a_k}|}{a_1!a_2!\cdots a_k!} = \binom{a_1 + a_2 + \cdots + a_k}{a_1, a_2, \ldots, a_k}.$$

Tracing from U_1 in the graph G, it is easy to see that all other paths of length $(m-1)$ ending at say U_{j+1} are of the form:

$$L_\sigma(\vec{a})U_pU_{p-1}\cdots U_{j+1},$$

where $1 \leqslant j < p \leqslant k, \vec{a} \in Q(k, m+j-p)$ and $\sigma \in S_{a_1+a_2+\cdots+a_k}$.
Therefore all non-zero terms in the expansion of $C(1)C(2)\cdots C(m)$ take the two forms:

1. $P_\sigma(\vec{a})$ where $\vec{a} \in Q(k, m)$ and each $\sigma \in S_{a_1+a_2+\cdots+a_k}$.
2. $P_\sigma(\vec{a})U_pU_{p-1}\cdots U_{j+1}$ for any j and p that $1 \leqslant j < p \leqslant k$, $\vec{a} \in Q(k, m+j-p)$ and $\sigma \in S_{a_1+a_2+\cdots+a_k}$.

Accounting for repeats and summing we have the following theorem:

Theorem 4.4. *The product $C(1)C(2)\cdots C(m)$ is given by:*

$$\begin{aligned} C(1)C(2)\cdots C(m) &= \sum_{\vec{a}\in Q(k,m)} \frac{1}{a_1!a_2!\cdots a_k!} \sum_{\sigma\in S_{a_1+\cdots+a_k}} P_\sigma(\vec{a}) \\ &+ \sum_{p=2}^{k}\sum_{j=1}^{p-1} \sum_{\vec{a}\in Q(k,m+j-p)} \frac{1}{a_1!a_2!\cdots a_k!} \sum_{\sigma\in S_{a_1+\cdots+a_k}} P_\sigma(\vec{a})U_pU_{p-1}\cdots U_{j+1}. \end{aligned} \tag{9}$$

Also the product $P_\sigma(\vec{a})$ with $\vec{a} \in Q(k, r)$ is:

$$\begin{aligned} P_\sigma(\vec{a}) &= \prod_{n=1}^{a_1+\cdots+a_k} \overline{U_{\chi_{\vec{a}}(\sigma(n))}}U_1\left(\sum_{s=1}^{n} \chi_{\vec{a}}(\sigma(s))\right) \\ &= \prod_{n=2}^{a_1+\cdots+a_k} c_{\chi_{\vec{a}}(\sigma(n))}\left(\sum_{s=1}^{n-1} \chi_{\vec{a}}(\sigma(s))\right)\cdot \overline{U_{\chi_{\vec{a}}(\sigma(1))}}U_1(r). \end{aligned} \tag{10}$$

Note that $\displaystyle\sum_{s=1}^{a_1+\cdots+a_k} \chi_{\vec{a}}(\sigma(s)) = \sum_{s=1}^{a_1+\cdots+a_k} \chi_{\vec{a}}(s) = r$ *as* $\sigma \in S_{a_1+\cdots+a_k}$.

We will also need a formula for the matrix product $C(i+1)C(i+2)\cdots C(m)$ for $1 \leqslant i \leqslant m$. To account for the translation by i, we define the product $P_\sigma(\vec{a}, i)$ for $\vec{a} \in Q(k, r)$:

$$\begin{aligned} P_\sigma(\vec{a}, i) &= \prod_{n=1}^{a_1+\cdots+a_k} \overline{U_{\chi_{\vec{a}}(\sigma(n))}}U_1\left(i+\sum_{s=1}^{n} \chi_{\vec{a}}(\sigma(s))\right) \\ &= \prod_{n=2}^{a_1+\cdots+a_k} c_{\chi_{\vec{a}}(\sigma(n))}\left(i+\sum_{s=1}^{n-1} \chi_{\vec{a}}(\sigma(s))\right)\cdot \overline{U_{\chi_{\vec{a}}(\sigma(1))}}U_1(r-i). \end{aligned} \tag{11}$$

Set $B(q) = C(i+q)$ for $1 \leqslant q \leqslant m-i$. Then $B(q)$ is the companion matrix associated to the vector-valued function $\vec{b}(q) = \vec{c}(i+q)$. We apply Theorem 4.4 to compute $B(1)B(2)\cdots B(m-i) = C(i+1)C(i+2)\cdots C(m)$. Replacing m by $m-i$, and translating $\vec{c}$ by i, we obtain the following corollary:

Corollary 4.5. *The product $C(i+1)C(i+2)\cdots C(m)$ is given by:*

$$C(i+1)\cdots C(m) = \sum_{\vec{a}\in Q(k,m-i)} \frac{1}{a_1!a_2!\cdots a_k!} \sum_{\sigma\in S_{a_1+\cdots+a_k}} P_\sigma(\vec{a}, i)$$
$$+\sum_{p=2}^{k}\sum_{r=1}^{p-1} \sum_{\vec{a}\in Q(k,m-i+j-p)} \frac{1}{a_1!a_2!\cdots a_k!} \sum_{\sigma\in S_{a_1+\cdots+a_k}} P_\sigma(\vec{a}, i)U_pU_{p-1}\cdots U_{j+1}. \tag{12}$$

5. Computing the entries of $C(0)C(1)C(2)\cdots C(m)$

We could now compute the (i,j)-entry m_{ij} of $C(0)C(1)C(2)\cdots C(m)$. It may appear that Theorem 4.4 could already give us the entry formula for a product of companion matrices but after some considerations one could see that notations are much more complicated if we look directly at $C(1)C(2)\cdots C(m)$. Interestingly, including $C(0)$ would help simplify our final formula for m_{ij}.
We recall the following matrices:

$$U_1(n) = \sum_{r=1}^{k} c_r(n)E_{1,r}; \tag{13}$$

$$\overline{U_i}U_1(n) = \sum_{j=1}^{k} c_j(n)E_{i,j}. \tag{14}$$

By Lemma 4.1, we also have:

$$U_1(n)U_pU_{p-1}\cdots U_{j+1} = \sum_{r=1}^{k} c_r(n)E_{1,r}E_{p,p-1}E_{p-1,p-2}\cdots E_{j+1,j} = c_p(n)E_{1,j}, \tag{15}$$

$$\overline{U_i}U_1(n)U_pU_{p-1}\cdots U_{j+1} = \sum_{r=1}^{k} c_r(n)E_{i,r}E_{p,p-1}E_{p-1,p-2}\cdots E_{j+1,j} = c_p(n)E_{i,j}. \tag{16}$$

For clarity purposes, we will compute m_{1j} first and then consider the general entry m_{ij}. Observe from Eqs. (13)–(16) that only the product terms beginning with U_1 and ending with U_1 or U_{j+1} would contribute to the $(1,j)$-entry of $C(0)C(1)C(2)\cdots C(m)$.

Now $C(0) = (U_1(0) + U_2 + \cdots + U_k)$. Then by the above observation and the sum formula for $C(1)C(2)\cdots C(m)$ in Theorem 4.4, the entry m_{1j} of $C(0)C(1)C(2)\cdots C(m)$ is supported by the matrices of the forms $U_1(0)P_\sigma(\vec{a})$ and $U_1(0)P_\sigma(\vec{a})U_pU_{p-1}\cdots U_{j+1}$ for $j+1 \leqslant p \leqslant k$. So we only need to collect the $(1,j)$-entry of the product $U_1(0)C(1)C(2)\cdots C(m)$ given by:

$$\sum_{\vec{a}\in Q(k,m)} \frac{1}{a_1!a_2!\cdots a_k!} \sum_{\sigma\in S_{a_1+\cdots+a_k}} U_1(0)P_\sigma(\vec{a})$$
$$+\sum_{p=j+1}^{k} \sum_{\vec{a}\in Q(k,m+j-p)} \frac{1}{a_1!a_2!\cdots a_k!} \sum_{\sigma\in S_{a_1+\cdots+a_k}} U_1(0)P_\sigma(\vec{a})U_pU_{p-1}\cdots U_{j+1} \tag{17}$$

Referring to the formula of $P_\sigma(\vec{a})$ in Theorem 4.4, we compute for $j \leqslant p \leqslant k$, the product $U_1(0)P_\sigma(\vec{a})$:

$$U_1(0)P_\sigma(\vec{a}) = \prod_{n=2}^{a_1+\cdots+a_k} c_{\chi_{\vec{a}}(\sigma(n))}\left(\sum_{s=1}^{n-1}\chi_{\vec{a}}(\sigma(s))\right)\cdot U_1(0)\overline{U_{\chi_{\vec{a}}(\sigma(1))}}U_1(m+j-p)$$

$$\begin{aligned} &= \prod_{n=2}^{a_1+\cdots+a_k} c_{\chi_{\vec{a}}(\sigma(n))}\left(\sum_{s=1}^{n-1}\chi_{\vec{a}}(\sigma(s))\right)\cdot c_{\chi_{\vec{a}}(\sigma(1))}(0)\cdot U_1(m+j-p) \\ &= \prod_{n=1}^{a_1+\cdots+a_k} c_{\chi_{\vec{a}}(\sigma(n))}\left(\sum_{s=1}^{n-1}\chi_{\vec{a}}(\sigma(s))\right)\cdot U_1(m+j-p). \end{aligned} \tag{18}$$

And also

$$\begin{aligned} U_1(0)P_\sigma(\vec{a})U_pU_{p-1}\cdots U_{j+1} &= \prod_{n=1}^{a_1+\cdots+a_k} c_{\chi_{\vec{a}}(\sigma(n))}\left(\sum_{s=1}^{n-1}\chi_{\vec{a}}(\sigma(s))\right)\cdot U_1(m+j-p)U_pU_{p-1}\cdots U_{j+1} \\ &= \prod_{n=1}^{a_1+\cdots+a_k} c_{\chi_{\vec{a}}(\sigma(n))}\left(\sum_{s=1}^{n-1}\chi_{\vec{a}}(\sigma(s))\right)\cdot c_p(m+j-p)E_{1j}. \end{aligned} \tag{19}$$

Using Eqs. (17)–(19), we obtain the $(1,j)$-entry of the matrix product $C(0)C(1)C(2)\cdots C(m)$:

$$\begin{aligned} m_{1j} = &\sum_{\vec{a}\in Q(k,m)} \frac{1}{a_1!a_2!\cdots a_k!} \\ &\times \sum_{\sigma\in S_{a_1+\cdots+a_k}} \prod_{n=1}^{a_1+\cdots+a_k} c_{\chi_{\vec{a}}(\sigma(n))}\left(\sum_{s=1}^{n-1}\chi_{\vec{a}}(\sigma(s))\right)\cdot c_j(m) \\ &+ \sum_{p=j+1}^{k}\sum_{\vec{a}\in Q(k,m+j-p)} \frac{1}{a_1!a_2!\cdots a_k!} \\ &\times \sum_{\sigma\in S_{a_1+\cdots+a_k}} \prod_{n=1}^{a_1+\cdots+a_k} c_{\chi_{\vec{a}}(\sigma(n))}\left(\sum_{s=1}^{n-1}\chi_{\vec{a}}(\sigma(s))\right)\cdot c_p(m+j-p) \\ = &\sum_{p=j}^{k} c_p(m+j-p)\cdot \sum_{\vec{a}\in Q(k,m+j-p)} \frac{1}{a_1!a_2!\cdots a_k!} \\ &\times \sum_{\sigma\in S_{a_1+\cdots+a_k}} \prod_{n=1}^{a_1+\cdots+a_k} c_{\chi_{\vec{a}}(\sigma(n))}\left(\sum_{s=1}^{n-1}\chi_{\vec{a}}(\sigma(s))\right). \end{aligned} \tag{20}$$

Recall $T_{k,m}$ in Section 2 that

$$T_{k,m} = \sum_{\vec{a}\in Q(k,m)} \frac{1}{a_1!a_2!\cdots a_k!} \sum_{\sigma\in S_{a_1+\cdots+a_k}} \prod_{n=1}^{a_1+\cdots+a_k} c_{\chi_{\vec{a}}(\sigma(n))}\left(\sum_{s=1}^{n-1}\chi_{\vec{a}}(\sigma(s))\right).$$

Therefore we have:

$$m_{1j} = \sum_{p=j}^{k} c_p(m+j-p)T_{k,m+j-p}. \tag{21}$$

To obtain m_{ij} for $2 \leqslant i \leqslant k$, we observe from Eqs. (13)–(16) that only the product terms beginning with U_i and ending with U_1 or U_{j+1} would contribute to the (i,j)-entry of $C(0)C(1)C(2)\cdots C(m)$. So we only need to collect the (i,j)-entry of the product

$$\overline{U_i}U_1(i-1)C(i)C(i+1)\cdots C(m).$$

By Corollary 4.5, $\overline{U_i}U_1(i-1)C(i)C(i+1)\cdots C(m)$ is given by:

$$\sum_{\vec{a}\in Q(k,m-i+1)} \frac{1}{a_1!a_2!\cdots a_k!} \sum_{\sigma\in S_{a_1+\cdots+a_k}} \overline{U_i}U_1(i-1)P_\sigma(\vec{a},i-1)$$
$$+\sum_{p=j+1}^{k} \sum_{\vec{a}\in Q(k,m-i+1+j-p)} \frac{1}{a_1!a_2!\cdots a_k!} \tag{22}$$
$$\times \sum_{\sigma\in S_{a_1+\cdots+a_k}} \overline{U_i}U_1(i-1)P_\sigma(\vec{a},i-1)U_pU_{p-1}\cdots U_{j+1}.$$

Referring to the formula of $P_\sigma(\vec{a},i-1)$ in Corollary 4.5, we compute for $j\leqslant p\leqslant k$, the products:

$$\begin{aligned}\overline{U_i}U_1(i-1)P_\sigma(\vec{a},i-1) &= \prod_{n=2}^{a_1+\cdots+a_k} c_{\chi_{\vec{a}}(\sigma(n))}\left(i-1+\sum_{s=1}^{n-1}\chi_{\vec{a}}(\sigma(s))\right)\\ &\quad\times \overline{U_i}U_1(i-1)\overline{U_{\chi_{\vec{a}}(\sigma(1))}}U_1(m-i+1+j-p)\\ &= \prod_{n=2}^{a_1+\cdots+a_k} c_{\chi_{\vec{a}}(\sigma(n))}\left(i-1+\sum_{s=1}^{n-1}\chi_{\vec{a}}(\sigma(s))\right)\\ &\quad\times \quad c_{\chi_{\vec{a}}(\sigma(1))}(i-1)\cdot U_1(m-i+1+j-p)\\ &= \prod_{n=1}^{a_1+\cdots+a_k} c_{\chi_{\vec{a}}(\sigma(n))}\left(i-1+\sum_{s=1}^{n-1}\chi_{\vec{a}}(\sigma(s))\right)\\ &\quad\cdot U_1(m-i+1+j-p)\end{aligned} \tag{23}$$

and

$$\begin{aligned}\overline{U_i}U_1(i-1)P_\sigma(\vec{a})U_pU_{p-1}\cdots U_{j+1} &= \prod_{n=1}^{a_1+\cdots+a_k} c_{\chi_{\vec{a}}(\sigma(n))}\left(i-1+\sum_{s=1}^{n-1}\chi_{\vec{a}}(\sigma(s))\right)\\ &\quad\times U_1(m-i+1+j-p)U_pU_{p-1}\cdots U_{j+1}\\ &= \prod_{n=1}^{a_1+\cdots+a_k} c_{\chi_{\vec{a}}(\sigma(n))}\left(i-1+\sum_{s=1}^{n-1}\chi_{\vec{a}}(\sigma(s))\right)\\ &\quad\cdot c_p(m-i+1+j-p)E_{ij}.\end{aligned} \tag{24}$$

Using Eqs. (22)–(24), we obtain the (i,j)-entry of the matrix product $C(0)C(1)C(2)\cdots C(m)$:

$$\begin{aligned}m_{ij} = &\sum_{\vec{a}\in Q(k,m-i+1)} \frac{1}{a_1!a_2!\cdots a_k!}\\ &\times \sum_{\sigma\in S_{a_1+\cdots+a_k}} \prod_{n=1}^{a_1+\cdots+a_k} c_{\chi_{\vec{a}}(\sigma(n))}\left(i-1+\sum_{s=1}^{n-1}\chi_{\vec{a}}(\sigma(s))\right)\cdot c_j(m-i+1)\\ &+\sum_{p=j+1}^{k}\sum_{\vec{a}\in Q(k,m-i+1+j-p)} \frac{1}{a_1!a_2!\cdots a_k!}\\ &\times \sum_{\sigma\in S_{a_1+\cdots+a_k}} \prod_{n=1}^{a_1+\cdots+a_k} c_{\chi_{\vec{a}}(\sigma(n))}\left(i-1+\sum_{s=1}^{n-1}\chi_{\vec{a}}(\sigma(s))\right)\cdot c_p(m-i+1+j-p).\end{aligned}$$

$$m_{ij} = \sum_{p=j}^{k} c_p(m-i+1+j-p) \cdot \sum_{\vec{a} \in Q(k,m-i+1+j-p)} \frac{1}{a_1!a_2! \cdots a_k!}$$

$$\times \sum_{\sigma \in S_{a_1+\cdots+a_k}} \prod_{n=1}^{a_1+\cdots+a_k} c_{\chi_{\vec{a}}(\sigma(n))} \left(i-1+\sum_{s=1}^{n-1} \chi_{\vec{a}}(\sigma(s)) \right)$$

$$= \sum_{p=j}^{k} c_p(m-i+1+j-p) T_{k,m-i+1+j-p}(i-1). \tag{25}$$

This proves Theorem 2.1.

6. The linear difference equation

We apply Theorem 2.1 to solve linear difference equations. Consider the kth order linear homogeneous difference equation:

$$X(m+k) = \alpha_1(m)X(m+k-1) + \alpha_2(m)X(m+k-2) + \cdots + \alpha_k(m)X(m). \tag{26}$$

Our equation is kth order only if $\alpha_k(m)$ is non-zero for some m. Without loss of generality, we assume that $\alpha_k(0)$ is non-zero throughout this exposition. Let $\{A(m)\}$ be the sequence of companion matrices associated to the function $\vec{\alpha}(m) = (\alpha_1(m), \alpha_2(m), \ldots, \alpha_k(m))$.

Let $\mathbf{B}(m) = (X(m+k-1), X(m+k-2), \ldots, X(m))^T$. Then rewriting Eq. (26) as a system of equations gives

$$\begin{bmatrix} X(m+k) \\ X(m+k-1) \\ \vdots \\ X(m+1) \end{bmatrix} = \begin{bmatrix} \alpha_1(m)X(m+k-1) + \cdots + \alpha_k(m)X(m) \\ X(m+k-1) \\ \vdots \\ X(m+1) \end{bmatrix}.$$

So $\mathbf{B}(m+1) = A(m) \cdot \mathbf{B}(m)$. Then $\mathbf{B}(m+1) = A(m)A(m-1) \cdots A(0) \cdot \mathbf{B}(0)$. Let m_{ij} be the (i,j)-entry of the product $A(m)A(m-1) \cdots A(0)$. Set $\mathbf{B}(0) = (0, \ldots, 0, 1, 0, \ldots, 0)^T$ where 1 is at the jth entry for $1 \leqslant j \leqslant k$. Then $\mathbf{B}(m+1) = (m_{1j}, m_{2j}, \ldots, m_{kj})^T$ so we have:

$$m_{1j} = X(m+k), \quad m_{2j} = X(m+k-1), \quad \ldots \quad, m_{kj} = X(m+1).$$

In general, consider the non-homogeneous linear difference equation:

$$X(m+k) = \alpha_1(m)X(m+k-1) + \alpha_2(m)X(m+k-2) + \cdots + \alpha_k(m)X(m) + f(m) \tag{27}$$

Set $\mathbf{F}(m) = (f(m), 0, \ldots, 0)^T$. Then writing as a system of equations, we have:

$$\mathbf{B}(m+1) = A(m) \cdot \mathbf{B}(m) + \mathbf{F}(m).$$

Iterating the equation we have:

$$\mathbf{B}(m+1) = \prod_{n=0}^{m} A(m-n) \cdot \mathbf{B}(0) + \sum_{q=0}^{m-1} \left[\prod_{n=0}^{q} A(m-n) \cdot \mathbf{F}(m-q-1) \right] + \mathbf{F}(m). \tag{28}$$

Using the product formula for companion matrix with $\vec{c}(j) = \vec{\alpha}(m-j)$ $(0 \leqslant j \leqslant m)$ in Theorem 2.1 yields solutions for the difference equations (26) and (27). Moreover, the first entry of the last two summands of (28) give a particular solution of Eq. (27). We summarize these observations in a corollary (of Theorem 2.1).

Corollary 6.1. *Set $\vec{c}(n) = \vec{\alpha}(m-n)$ for $0 \leqslant n \leqslant m$. A basis for the solution space of Eq. (26) is given by, $1 \leqslant j \leqslant k$,*

$$\Theta_j(m+k) = \sum_{p=j}^{k} \alpha_p(p-j)T_{k,m+j-p}.$$

Moreover, a particular solution of Eq. (27) is given by

$$\Phi(m+k) = f(m) + \sum_{q=0}^{m-1}\sum_{p=1}^{k} \alpha_p(m-q-1+p)T_{k,q+1-p}f(m-q-1).$$

Next, we make some interesting observations about the solution space of Eq. (26). For $\mathbf{B}(0) = (0,\ldots,0,1)^T$, the solution of Eq. (26) is $\Theta_k(m+k) = \alpha_k(0)T_{k,m}$. Moreover, we also have $\mathbf{B}(1) = (\alpha_k(0),0,\ldots,0)^T = \alpha_k(0)\cdot(1,0,\ldots,0)^T$. Note that $\alpha_k(0)$ is non-zero and that $(1,0,\ldots,0)^T$ is the initial condition for solution $\Theta_1(m+k)$. Therefore by a shift of index we have

$$\Theta_1(m+k) = \frac{1}{\alpha_k(0)}\Theta_k(m+k+1) = T_{k,m+1}.$$

By Corollary 6.1, $T_{k,m+1} = \sum_{p=1}^{k} \alpha_p(p-1)T_{k,m+1-p}$ for $m \geqslant 0$.

Moreover, the formula for $\Theta_j(m+k)$ gives:

$$\begin{bmatrix} \Theta_1(m+k) \\ \Theta_2(m+k) \\ \vdots \\ \Theta_k(m+k) \end{bmatrix} = \begin{bmatrix} \alpha_k(k-1) & \alpha_{k-1}(k-2) & \ldots & \alpha_2(1) & \alpha_1(0) \\ 0 & \alpha_k(k-2) & \ldots & \alpha_3(1) & \alpha_2(0) \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & \alpha_k(0) \end{bmatrix} \begin{bmatrix} T_{k,m-k+1} \\ T_{k,m-k+2} \\ \vdots \\ T_{k,m} \end{bmatrix}.$$

If $\alpha_k(j) \neq 0$ for all $0 \leqslant j \leqslant k-1$, the above equation gives us a change of basis relation.

Corollary 6.2. *For $\vec{c}(j) = \vec{\alpha}(m-j)$,*

$$T_{k,m+1} = \sum_{p=1}^{k} \alpha_p(p-1)T_{k,m+1-p}.$$

Moreover, if $\alpha_k(j) \neq 0$ for $j = 0, 1, \ldots, k-1$, the set

$$\{T_{k,m-j};\, j = 0, 1, \ldots, k-1\}$$

forms a basis for the solution space of Eq. (26).

Observe that if $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants then, using the fact that $|S_n| = n!$ and collecting like terms, we have

$$T_{k,m+1} = \sum_{(a_1,a_2,\ldots,a_k)\in Q(k,m+1)} \binom{a_1+a_2+\cdots+a_k}{a_1,a_2,\ldots,a_k} \alpha_1^{a_1}\alpha_2^{a_2}\cdots\alpha_k^{a_k}.$$

This is the solution Θ_1 of Eq. (26) with constant coefficients and initial conditions $X(0) = 0 = \cdots = X(k-2)$ and $X(k-1) = 1$. The same answer is obtained in [2] and is listed as Identity 5. In the same paper, $T_{k,m+1}$ is called the generalized kth order Fibonacci number and is obtained by counting the number of ways to tile a board of length $(m+1)$ with colored tiles of length at most k.

7. Conclusions

We see in this paper that the entries in the product of companion matrices could be expressed in terms of a single function $T_{k,m}$. In particular, it is interesting to note that the solution space of Eq. (26) is spanned by the integer translates of the function $T_{k,m}$ as noted in Corollary 6.2. As further work, we could analyze $T_{k,m}$ to understand qualitative properties of the solutions of linear difference equation. It would also be interesting to explore applications of the product formula to Markov chains and random walks.

Another direction to explore is, perhaps, the further extension of the weight-free graph idea in this paper to other combinatorial matrix analysis in [1] and non-homogeneous product of matrices discussed in [8].

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