The H-Principle

Topics in Topology II
Math 80440, Spring 2018

Instructor: Christopher Schommer-Pries
Office: 202 Hayes Healy Hall
email: cschomme@nd.edu

Course Description:
The *Homotopy Principle* (or *h-principle*) is a method to reduce certain problems in differential geometry to homotopy-theoretic problems. What kind of problems? Here are some examples (1) Can you turn a sphere inside out? (2) Does a given manifold M admit a Riemannian metric of positive or negative curvature? (3) Does it admit a symplectic structure? (4) Can we immerse a Riemannian manifold into Euclidean space isometrically? (5) Does M admit a foliation of a given codimension? (6) Can we classify submersions $M \to W$ up to regular homotopy?

All these structures can be described as sections of a certain fiber bundle on M that obey a certain partial differential equation or partial differential inequality (Problems 1,2,6). If the manifold is *open*, i.e. has no closed connected component, then a very general theorem of Gromov reduces solving a partial differential inequality of the above type to a purely homotopy theoretic problem. In some cases, for example Smale's work on sphere eversions, the methods apply to closed manifolds as well.

The h-principle has been proven in a much more general framework and has been successfully applied to problems which do not look even remotely related to differential inequalities. Some examples are:

- McDuff's work on the homology of configuration spaces;
- Galatius' work on the stable homology of the automorphism group of free groups.
- New proofs of the Galatius-Madsen-Tillmann-Weiss theorem identifying the classifying space of the cobordism category with a certain Thom spectrum.
- Igusa's space of framed generalize Morse functions is contractible, a key step in the proof of the cobordism hypothesis.
- Non-abelian Poincaré duality and the Dold-Thom theorem.

This course is dedicated to understanding the homotopy principle. What precisely is the homotopy principle? When can we use it? How do we do so? There will be many examples ranging over topology and differential geometry. We will also learn the secret beauty behind the letter 'S'.

Prerequisites: The basic topology and geometry sequences (Math 60330, 60440, 60670) and intermediate geometry and topology (Math 70330). In particular I will assume familiarity with homology, cohomology, vector and fiber bundles, smooth manifolds, including tangent bundles, Riemannian manifolds, and characteristic classes. Familiarity with basic category theory and the formalism of sheaves will be useful.