

Bounds on the Pseudo-Weight of Minimal Pseudo-Codewords of Projective Geometry Codes

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ABSTRACT. In this paper we focus our attention on a family of finite geometry codes, called type-I projective geometry low-density parity-check (PG-LDPC) codes, that are constructed based on the projective planes $\text{PG}(2, q)$. In particular, we study their minimal codewords and pseudo-codewords, as it is known that these vectors characterize completely the code performance under maximum-likelihood decoding and linear programming decoding, respectively. The main results of this paper consist of upper and lower bounds on the pseudo-weight of the minimal pseudo-codewords of type-I PG-LDPC codes.

1. Introduction

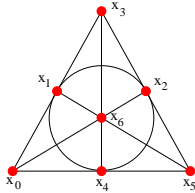
The family of type-I PG-LDPC codes that we study here are cyclic codes which moreover have very concise descriptions and large automorphism groups. It has been observed experimentally that these codes have a good performance, close to maximum-likelihood decoding, if decoded with iterative decoding or linear programming decoding, see e.g. [1, 2, 3, 4]. Hence these codes are worthwhile study objects and this paper aims at quantifying the difference between maximum-likelihood decoding and linear programming decoding of these codes.

The codes under consideration are defined as follows. Let $q \triangleq 2^s$ for some positive integer s , and consider a finite projective plane $\text{PG}(2, q)$ (see e.g. [5]) with $q^2 + q + 1$ points and $q^2 + q + 1$ lines: each point lies on $q + 1$ lines and each line contains $q + 1$ points. A standard way of associating a parity-check matrix \mathbf{H} of a binary linear code to a finite geometry is to let the set of points correspond to the columns of \mathbf{H} , to let the set of lines correspond to the rows of \mathbf{H} , and finally to define the entries of \mathbf{H} according to the incidence structure of the finite geometry. In this way, we can associate to the projective plane $\text{PG}(2, q)$ the code $\mathcal{C}_{\text{PG}(2, q)}$ with parity-check matrix $\mathbf{H} \triangleq \mathbf{H}_{\text{PG}(2, q)}$, whose parameters are $[q^2 + q + 1, n - 3^s - 1, q + 2]$. This code has the nice property that, with an appropriate ordering of the columns and rows, the parity-check matrix is a circulant matrix, meaning that $\mathcal{C}_{\text{PG}(2, q)}$ is a cyclic code.

EXAMPLE 1.1. As an example, we take $q = 2$ to obtain the Fano plane as in Figure 1.1 with a labeling that leads to a 7×7 circulant matrix $\mathbf{H}_{\text{PG}(2, 2)}$, and hence

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FIGURE 1. PG(2, 2), with a labeling leading to a cyclic code



to a $[7, 3, 4]$ binary cyclic code as its nullspace:

$$H_{\text{PG}(2,2)} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The paper is structured as follows. In Section 2 we introduce the framework necessary for the development of our results. We will talk about maximum likelihood (ML) and linear programming (LP) decoding, and introduce our main objects of study, pseudo-codewords, which are vectors that appear in connection to iterative and linear program decoding. Upper and lower bounds on the pseudo-weight of minimal pseudo-codewords will be given in Section 3, which contains the new results of this paper. In the last section we offer some conclusions.

2. Background and Definitions

Consider a binary linear code \mathcal{C} of length n and dimension k that is used for data communication over a memoryless binary-input channel. The codeword that is transmitted will be called \mathbf{x} whereas the received vector will be called \mathbf{y} . Based on the received vector, if we define the log-likelihood ratios (LLR) to be $\lambda_i \triangleq \log(p_{Y_i|X_i}(y_i|0)/p_{Y_i|X_i}(y_i|1))$, $i = 1, \dots, n$, we can view ML decoding as the optimization problem

$$(2.1) \quad \hat{\mathbf{x}} \triangleq \arg \min_{\mathbf{x} \in \mathcal{C}} \sum_{i=1}^n x_i \lambda_i = \arg \min_{\mathbf{x} \in \text{conv}(\mathcal{C})} \sum_{i=1}^n x_i \lambda_i,$$

where $\text{conv}(\mathcal{C})$ is the convex hull of \mathcal{C} in \mathbb{R}^n , see [3, 4, 6, 7].

Among the codewords, the significant ones for ML decoding turn out to be the *minimal* ones, $\mathcal{M}(\mathcal{C})$, i.e. the ones with support not containing the support of any other non-zero codeword as proper subset. Indeed, since we use a binary linear code \mathcal{C} over a binary-input output-symmetric channel, we can without loss of generality assume that the zero codeword was sent, because all decision regions are congruent. Then we have that, if $\mathbf{x} \in \mathcal{C}$, the region $\mathcal{D}_{\mathbf{x}}^{\text{ML}}$ in the LLR space where the ML decoder decides in favor of the codeword \mathbf{x} , $\mathcal{D}_{\mathbf{x}}^{\text{ML}} \triangleq \{\boldsymbol{\lambda} \in \mathbb{R}^n \mid \mathbf{x}' \cdot \boldsymbol{\lambda}^{\text{T}} \geq \mathbf{x}' \cdot \boldsymbol{\lambda}^{\text{T}} \text{ for all } \mathbf{x}' \in \mathcal{C} \setminus \{\mathbf{x}\}\}$,¹ shares a facet with the decision region $\mathcal{D}_{\mathbf{0}}^{\text{ML}}$ of the

¹Note that during ML decoding, ties between decoding regions can either be resolved in a random or in a systematic fashion.

zero codeword if and only if $\mathbf{x} \in \mathcal{M}(\mathcal{C})$. Hence the set of minimal codewords are of particular interest and are worth studying, as their patterns characterize the behavior of the code under ML decoding.

For most codes of interest, the description complexity of $\text{conv}(\mathcal{C})$ grows exponentially in the block length, therefore finding the minimum in (2.1) with a linear programming solver is highly impractical for reasonably long codes. Hence we relax the problem by replacing the above minimization by a minimization over some easily describable polytope \mathcal{P} that is a relaxation of $\text{conv}(\mathcal{C})$, i.e.

$$(2.2) \quad \hat{\mathbf{x}} \triangleq \arg \min_{\mathbf{x} \in \mathcal{P}} \sum_{i=1}^n x_i \lambda_i.$$

A relaxation commonly used, see [6, 7, 8], is given by *the fundamental polytope* \mathcal{P} defined as

$$\mathcal{P} \triangleq \bigcap_{i=1}^m \text{conv}(\mathcal{C}_i) \quad \text{with} \quad \mathcal{C}_i \triangleq \{\mathbf{x} \in \{0, 1\}^n \mid \mathbf{r}_i \mathbf{x}^T = 0 \pmod{2}\},$$

where $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m$ are the rows of a parity-check matrix \mathbf{H} . Points in the set \mathcal{P} are called *pseudo-codewords*. Because the set \mathcal{P} is usually strictly larger than $\text{conv}(\mathcal{C})$, it can obviously happen that the decoding rule in (2.2) delivers a vertex of \mathcal{P} that is not a codeword. Such vertices that correspond to pseudo-codewords that are not codewords are the reason for the sub-optimality of LP decoding (cf. [7, 8]). Note that $\mathcal{P} = \mathcal{P}(\mathbf{H})$ is a function of the parity-check matrix \mathbf{H} that describes the code \mathcal{C} ; different parity-check matrices for the same code might therefore lead to different fundamental polytopes.

For LP decoding of a binary linear code that is used for data transmission over a binary-input output-symmetric channel, it is sufficient to consider the *fundamental cone* $\mathcal{K}(\mathbf{H})$, which is the part of the fundamental polytope \mathcal{P} around the vertex $\mathbf{0}$ and stretched to infinity. This fundamental cone can be characterized as follows.

LEMMA 2.1 ([6, 7, 8]). *Let \mathcal{C} be an arbitrary binary linear code and let \mathbf{H} be its $m \times n$ parity-check matrix. Let $\mathbf{r}_1, \dots, \mathbf{r}_m$ be the row vectors of \mathbf{H} . For each row vector \mathbf{r}_j , $j = \overline{1, m}$, we let $\text{supp}(\mathbf{r}_j)$ denote its support in $\{1, \dots, n\}$.*

Then, the fundamental cone $\mathcal{K}(\mathbf{H})$ is the set of vectors $\boldsymbol{\omega} \in \mathbb{R}^n$ with $\omega_i \geq 0$, $i = \overline{1, n}$, such that for each $j \in \{1, \dots, m\}$ and for each $i \in \text{supp}(\mathbf{r}_j)$,

$$\sum_{i' \in \text{supp}(\mathbf{r}_j) \setminus \{i\}} \omega_{i'} \geq \omega_i.$$

Any point in $\mathcal{K}(\mathbf{H})$ will also be called a pseudo-codeword. We note that if $\boldsymbol{\omega} \in \mathcal{K}(\mathbf{H})$, then also $\alpha \cdot \boldsymbol{\omega} \in \mathcal{K}(\mathbf{H})$ for any $\alpha > 0$, and so we will say that two such pseudo-codewords are in the same equivalence class. Moreover, for any $\boldsymbol{\omega} \in \mathcal{K}(\mathbf{H})$ there exists an $\alpha > 0$ (in fact, a whole interval of α 's) such that $\alpha \cdot \boldsymbol{\omega} \in \mathcal{P}(\mathbf{H})$. Clearly, pseudo-codewords are not codewords in general. As before, if we let $\mathcal{D}_0^{\text{LP}} \triangleq \{\boldsymbol{\lambda} \in \mathbb{R}^n \mid \boldsymbol{\omega} \cdot \boldsymbol{\lambda}^T \geq 0 \text{ for all } \boldsymbol{\omega} \in \mathcal{P}(\mathbf{H}) \setminus \{\mathbf{0}\}\}$ be the region where the LP decoder decides in favor of the codeword $\mathbf{0}$, it can easily be seen that $\mathcal{D}_0^{\text{LP}} = \{\boldsymbol{\lambda} \in \mathbb{R}^n \mid \boldsymbol{\omega} \cdot \boldsymbol{\lambda}^T \geq 0 \text{ for all } \boldsymbol{\omega} \in \mathcal{K}(\mathbf{H})\}$.² Moreover, we have that $\mathcal{D}_0^{\text{LP}} = \{\boldsymbol{\lambda} \in \mathbb{R}^n \mid \boldsymbol{\omega} \cdot \boldsymbol{\lambda}^T \geq 0 \text{ for all } \boldsymbol{\omega} \in \mathcal{M}_p(\mathcal{K}(\mathbf{H}))\}$ where $\mathcal{M}_p(\mathcal{K}(\mathbf{H}))$ is the set of all vectors $\boldsymbol{\omega} \in \mathcal{K}(\mathbf{H})$ such

²Note that during LP decoding, ties between decoding regions can either be resolved in a random or in a systematic fashion.

that the set $\{\alpha \cdot \boldsymbol{\omega} \mid \alpha \geq 0\}$ is an edge of $\mathcal{K}(\mathbf{H})$. Mimicking the parallel ML discussion above, we will call these *minimal pseudo-codewords*. Therefore, the set $\mathcal{M}_p(\mathcal{K}(\mathbf{H}))$ completely characterizes the behavior of the LP decoder.

EXAMPLE 2.2. Continuing Ex. 1.1, the fundamental cone for $\mathbf{H} = \mathbf{H}_{\text{PG}(2,2)}$ is given by the set

$$\begin{aligned} \mathcal{K}(\mathbf{H}) = \{ & \boldsymbol{\omega} \in \mathbb{R}^7 \mid \omega_i \geq 0, i = \overline{1, 7}, \\ & -\omega_0 + \omega_1 + \omega_3 \geq 0, \quad +\omega_0 - \omega_1 + \omega_3 \geq 0, \quad +\omega_0 + \omega_1 - \omega_3 \geq 0, \\ & -\omega_1 + \omega_2 + \omega_4 \geq 0, \quad +\omega_1 - \omega_2 + \omega_4 \geq 0, \quad +\omega_1 + \omega_2 - \omega_4 \geq 0, \\ & -\omega_2 + \omega_3 + \omega_5 \geq 0, \quad +\omega_2 - \omega_3 + \omega_5 \geq 0, \quad +\omega_2 + \omega_3 - \omega_5 \geq 0, \\ & -\omega_3 + \omega_4 + \omega_6 \geq 0, \quad +\omega_3 - \omega_4 + \omega_6 \geq 0, \quad +\omega_3 + \omega_4 - \omega_6 \geq 0, \\ & -\omega_4 + \omega_5 + \omega_0 \geq 0, \quad +\omega_4 - \omega_5 + \omega_0 \geq 0, \quad +\omega_4 + \omega_5 - \omega_0 \geq 0, \\ & -\omega_5 + \omega_6 + \omega_1 \geq 0, \quad +\omega_5 - \omega_6 + \omega_1 \geq 0, \quad +\omega_5 + \omega_6 - \omega_1 \geq 0, \\ & -\omega_6 + \omega_0 + \omega_2 \geq 0, \quad +\omega_6 - \omega_0 + \omega_2 \geq 0, \quad +\omega_6 + \omega_0 - \omega_2 \geq 0\}. \end{aligned}$$

Among these pseudo-codewords, the minimal ones are the ones with the number of inequalities satisfied with equality larger than or equal to $n - 1 = 7 - 1$, and the rank of the matrix formed by the coefficients of these equalities is required to be $n - 1 = 7 - 1$.

REMARK 2.3. One can show that all minimal pseudo-codewords can be scaled such that all entries are non-negative integers. A word a caution: a pseudo-codeword $\boldsymbol{\omega}$ reduced modulo 2 might not be a codeword. However, there exists obviously a constant α (any even number would work) such that $\alpha \boldsymbol{\omega}$ reduced modulo 2 is a codeword. Throughout this paper we consider pseudo-codewords that have the smallest possible integer entries among all pseudo-codewords $\alpha \boldsymbol{\omega}$ with $\alpha > 0$, such that they are codewords if reduced modulo 2.

In the following we will only consider the additive white Gaussian noise channel (AWGNC). In this context, we will define an important parameter in LP decoding: the pseudo-weight of a pseudo-codeword, which corresponds to the Hamming weight of a codeword in ML decoding. Indeed, the significance of the Hamming weight $w_{\mathbf{H}}(\mathbf{x})$ of a minimal codeword \mathbf{x} for ML decoding is the following: it can be shown that the squared Euclidean distance from the point $+\mathbf{1}$ in signal space, corresponding to the codeword $\mathbf{0}$, to the boundary plane $\{\boldsymbol{\lambda} \in \mathbb{R}^n \mid \mathbf{x} \cdot \boldsymbol{\lambda}^{\text{T}} = 0\}$ of the decision region of $\mathbf{0}$ under ML decoding is $w_{\mathbf{H}}(\mathbf{x})$. Similarly, the AWGNC pseudo-weight is defined such that if $\boldsymbol{\omega}$ is a minimal pseudo-codeword then the squared Euclidean distance from the point $+\mathbf{1}$ in signal space, corresponding to the codeword $\mathbf{0}$, to the boundary plane $\{\boldsymbol{\lambda} \in \mathbb{R}^n \mid \boldsymbol{\omega} \cdot \boldsymbol{\lambda}^{\text{T}} = 0\}$ of the decision region of $\mathbf{0}$ under LP decoding is $w_p^{\text{AWGNC}}(\boldsymbol{\omega})$.

LEMMA 2.4 ([9, 8]). *The AWGNC pseudo-weight of a pseudo-codeword $\boldsymbol{\omega} \in \mathcal{K}(\mathbf{H})$ is equal to $w_p^{\text{AWGNC}}(\boldsymbol{\omega}) = \|\boldsymbol{\omega}\|_1^2 / \|\boldsymbol{\omega}\|_2^2$, where $\|\boldsymbol{\omega}\|_1$ and $\|\boldsymbol{\omega}\|_2$ are the 1-norm and 2-norm of $\boldsymbol{\omega}$.*³ \square

³Note that for $\mathbf{x} \in \{0, 1\}^n$ we have $w_p^{\text{AWGNC}}(\mathbf{x}) = w_{\mathbf{H}}(\mathbf{x})$, where $w_{\mathbf{H}}(\mathbf{x})$ is the Hamming weight of \mathbf{x} .

For smoother notations we will use $w_p(\boldsymbol{\omega})$ in place of $w_p^{\text{AWGNC}}(\boldsymbol{\omega})$. Obviously any two pseudo-codewords of the same equivalence class have the same pseudo-weight. Also, the minimum pseudo-weight is always upper bounded by the minimum distance of the code. Hence, in order to assess the performance under LP decoding, we need to look at the minimum pseudo-weight over the set \mathcal{S} of all minimal pseudo-codewords that are not multiples of minimal codewords. In particular we look at the *pseudo-weight spectrum gap* $g(\mathbf{H})$:

$$g(\mathbf{H}) \triangleq \min_{\boldsymbol{\omega} \in \mathcal{S}} w_p(\boldsymbol{\omega}) - d_{\min}.$$

For a randomly constructed code, the pseudo-weight spectrum gap can be negative. However for the $\text{PG}(2, q)$ - and $\text{EG}(2, q)$ -based codes, the pseudo-weight spectrum gap is *non-negative*, which reflects the good performance, close to maximum-likelihood, of these codes, under LP decoding. In fact, for the codes we investigated, the pseudo-weight spectrum gap is significantly positive.

EXAMPLE 2.5. In [3, 4] we were able to classify the minimal codewords of projective and Euclidean plane codes, with small q . We show here the results for $q = 2$ and $q = 4$. If $q = 2$ we obtain a $[7, 3, 4]$ code with 8 codewords, all of which are minimal. These are: $(0, 0, 1, 0, 1, 1, 1)$, $(1, 0, 0, 1, 0, 1, 1)$, $(1, 1, 0, 0, 1, 0, 1)$, $(1, 1, 1, 0, 0, 1, 0)$, $(0, 1, 1, 1, 0, 0, 1)$, $(1, 0, 1, 1, 1, 0, 0)$, $(0, 1, 0, 1, 1, 1, 0)$. It can be shown that these minimal codewords together with the following list of pseudo-codewords are the only minimal pseudo-codewords of this code. The list of non-codewords minimal pseudo-codewords are: $(2, 2, 1, 2, 1, 1, 1)$, $(1, 2, 2, 1, 2, 1, 1)$, $(1, 1, 2, 2, 1, 2, 1)$, $(1, 1, 1, 2, 2, 1, 2)$, $(2, 1, 1, 1, 2, 2, 1)$, $(1, 2, 1, 1, 1, 2, 2)$, $(2, 1, 2, 1, 1, 1, 2)$. The AWGNC pseudo-weight of these pseudo-codewords is 6.25, so we obtain a strictly positive gap of 2.25. In the case of the projective plane over \mathbb{F}_q where $q = 4$, we have a $[21, 11, 6]$ code with the set of minimal codewords given by all codewords of weight 6, 8, and 10. Moreover, one can show that the gap is $g(\mathbf{H}_{\text{PG}(2,4)}) = 9.8 - 6 = 3.8$, again strictly positive.⁴ The next code based on $\text{PG}(2, 8)$ behaves similarly. For details we refer to [3, 4].

3. Bounds on the Pseudo-Weight of Pseudo-Codewords

In this section we present results on projective-plane-based codes and their minimum pseudo-weights. In the following, q will be a power of 2, and $n = q^2 + q + 1$.

LEMMA 3.1. *Let \mathcal{C} be a code from projective plane over \mathbb{F}_q , with $\mathbf{H} \triangleq \mathbf{H}_{\text{PG}(2,q)}$ an $n \times n$ circulant parity-check matrix. Let $\boldsymbol{\omega}$ be a non-zero pseudo-codeword. For any $l \in \{1, 2, \dots, n\}$ we have*

$$\sum_{\substack{i=1 \\ i \neq l}}^n \omega_i \geq (q+1)\omega_l,$$

or, equivalently,

$$\sum_{i=1}^n \omega_i \geq (q+2)\omega_l.$$

⁴This result was obtained by simplifying the problem by using symmetries and then doing some brute-force computations with the help of “lrs” [10].

PROOF. We use the notation of Lemma 2.1. Let P be a point in the projective plane $\text{PG}(2, q)$ that has the associated pseudo-codeword component value ω_l . There are $q+1$ lines $\mathbf{r}_{j_i}, i \in \{1, \dots, q+1\}$ crossing P . For each line \mathbf{r}_{j_i} among the equations characterizing the fundamental cone $\mathcal{K}(\mathbf{H}), i \in \{1, \dots, q+1\}$, we find:

$$\sum_{i' \in \text{supp}(\mathbf{r}_{j_i}) \setminus \{l\}} \omega_{i'} \geq \omega_l.$$

Because the sets $\mathbf{r}_{j_i} \setminus \{P\}, i \in \{1, \dots, q+1\}$, partition the set of all points minus P , we obtain the desired inequalities by adding these $q+1$ equations. \square

The following theorem gives upper and lower bounds on the pseudo-weight of all minimal pseudo-codewords. The two bounds have been already derived by several other authors, e.g. [11], however the proof we present here makes use of some very simple arithmetic, and therefore we include it as well.

THEOREM 3.2. *Let \mathcal{C} be a code from projective plane over \mathbb{F}_q with $\mathbf{H}_{\text{PG}(2,q)}$ an $n \times n$ circulant parity-check matrix. Let $\boldsymbol{\omega}$ be a nonzero pseudo-codeword. Then*

$$q+2 \leq w_p(\boldsymbol{\omega}) \leq |\text{supp}(\boldsymbol{\omega})|.$$

PROOF. We have that

$$\sum_{l=1}^n \omega_l \sum_{\substack{i=1 \\ i \neq l}}^n \omega_i = \|\boldsymbol{\omega}\|_1^2 - \|\boldsymbol{\omega}\|_2^2 \quad \Rightarrow \quad w_p(\boldsymbol{\omega}) = 1 + \frac{\sum_{l=1}^n \omega_l \left(\sum_{\substack{i=1 \\ i \neq l}}^n \omega_i \right)}{\|\boldsymbol{\omega}\|_2^2}.$$

Using Lemma 3.1, i.e. $\sum_{\substack{l=i \\ i \neq l}}^n \omega_i \geq (q+1)\omega_l$, we obtain the lower bound $w_p(\boldsymbol{\omega}) \geq q+2$.

The upper bound is obtained using the Cauchy-Schwarz inequality. Indeed, the desired upper bound follows easily from

$$\begin{aligned} \left(\sum_{i=1}^n \omega_i \right)^2 &= \left(\sum_{i \in \text{supp}(\boldsymbol{\omega})} 1 \cdot \omega_i \right)^2 \leq \left(\sum_{i \in \text{supp}(\boldsymbol{\omega})} 1^2 \right) \left(\sum_{i \in \text{supp}(\boldsymbol{\omega})} \omega_i^2 \right) \\ &= |\text{supp}(\boldsymbol{\omega})| \cdot \left(\sum_{i \in \text{supp}(\boldsymbol{\omega})} \omega_i^2 \right). \end{aligned}$$

\square

Let now $w_p^{\min}(\mathbf{H})$ be the minimum AWGNC pseudo-weight of a code with parity-check matrix \mathbf{H} . Since the above lower bound matches the minimum Hamming weight, the following observation is immediate.

COROLLARY 3.3. *The minimum pseudo-weight of a projective plane code is $w_p^{\min}(\boldsymbol{\omega}) = q+2 = d_{\min}$.*

The upper bound in Th. 3.2 is obviously attained for codewords. However, for minimal pseudo-codewords that are not multiples of codewords, this upper bound seems to be the further away from the actual weight, the larger the bound. We are interested in these vectors, especially those with small pseudo-weight, because they are relevant in quantifying the spectrum gap, which in turn quantifies the performance difference between LP and ML decoding. Hence, we would like to give certain

estimates on the possible minimum pseudo-weight of such vectors. Our observation, made by studying the behavior of vectors in certain examples, was that pseudo-codewords of small support are candidates for small weight pseudo-codewords, and among these, the ones with values of 0, 1, and 2 for the entries. In the remaining of this section we will give upper and lower bounds on the pseudo-weight of vectors with integer entries, and on minimal pseudo-codewords in particular, with a special emphasis on the ones with entries equal to 0, 1, or 2.

THEOREM 3.4 (Conjecture). *The smallest weight among all minimal pseudo-codewords that are not codewords is given by the pseudo-codewords that contain only zeros, ones, and twos.*

3.1. Minimization of the Pseudo-Weight of a Vector. We would like to mention that the results that we present in this subsection were derived in by the second author in [12]. The main result is presented in Theorem 3.7 and consists of a lower bound on the pseudo-weight of a vector. Since this theorem does not use any property of a pseudo-codeword, it holds for any vector \mathbf{x} that has non-negative entries. In the following we will use the notation \mathbf{x} to identify an arbitrary vector with non-negative entries and $\boldsymbol{\omega}$ to identify a pseudo-codeword. Before presenting this theorem, we study first the influence of 0-components on the pseudo-weight.

LEMMA 3.5. *Let \mathbf{x} be a vector of length n with r 0-components and $n - r$ positive entries x_i where $r < n - 1$. Then there exists a vector \mathbf{x}' of length n with $n - r - 1$ positive entries such that $w_p(\mathbf{x}') < w_p(\mathbf{x})$.*

PROOF. First we multiply \mathbf{x} by an α such that the smallest non-zero component in $\tilde{\mathbf{x}} = \alpha\mathbf{x}$ is 1. The vector \mathbf{x}' is obtained by switching this 1-component \tilde{x}_l in $\tilde{\mathbf{x}}$ to 0. Let $s \triangleq \sum_{\substack{i=1 \\ i \neq l}}^n \tilde{x}_i$ and $t \triangleq \sum_{\substack{i=1 \\ i \neq l}}^n \tilde{x}_i^2$. From the assumptions in the lemma statement, it follows that $s \geq 1$ and $t \geq 1$. Moreover it holds that $s \leq t$. Therefore $1 + \frac{1}{t} \leq 1 + \frac{1}{s}$ and because $1 + \frac{1}{s} > 1$ we have that $1 + \frac{1}{t} < (1 + \frac{1}{s})^2$ or equivalently, $\frac{s^2}{t} < \frac{(s+1)^2}{t+1}$. Then:

$$w_p(\mathbf{x}') = \frac{s^2}{t} < \frac{(s+1)^2}{t+1} = w_p(\tilde{\mathbf{x}}) = w_p(\mathbf{x}).$$

□

To get prepared for the main result of this subsection we next show that the minimal pseudo-weight is obtained for vectors with a certain structure, namely vectors that contain only two different components. More precisely, we will prove that among all non-zero vectors with non-negative integer components in an interval $[m, M]$, $0 < m \leq M$, and with at least one component equal to m and one equal to M , the vectors with minimal pseudo-weight are the ones that have the non-zero components equal to either m or M .

LEMMA 3.6. *Let $0 < m \leq M$. Among all vectors $\mathbf{x} = (x_1, \dots, x_n)$ with $x_j = 0$ or $m \leq x_j \leq M$, $j \in \{1, \dots, n\}$, and with at least one $x_{j_1} = m$ and one $x_{j_2} = M$, the vectors with minimal pseudo-weight are the ones that have all non-zero components equal to either m or M .*

PROOF. Let $\mathbf{x} = (x_1, \dots, x_n)$ be a vector with the properties of the lemma statement. Suppose that \mathbf{x} has a components x_j with $m < x_j < M$. Let $A \triangleq$

$\sum_{i=1, i \neq j}^n x_i$ and $B \triangleq \sum_{i=1, i \neq j}^n x_i^2$. The pseudo-weight of \mathbf{x} is

$$w_p(\mathbf{x}) = \frac{(A + x_j)^2}{B + x_j^2}.$$

The first and second partial derivative of the pseudo-weight as a function of x_j are, respectively,

$$\begin{aligned} \frac{\partial w_p}{\partial x_j} &= -\frac{2(A + x_j)(Ax_j - B)}{(B + x_j^2)^2}, \\ \frac{\partial^2 w_p}{\partial x_j^2} &= \frac{2(B^2 - 3Bx_j^2 - 6ABx_j + 2Ax_j^3 + 3A^2x_j^2 - A^2B)}{(B + x_j^2)^3}. \end{aligned}$$

The critical points of the pseudo-weight function are therefore $x_{j,1} = -A < 0$ and $x_{j,2} = \frac{B}{A} > 0$. With $\frac{\partial^2 w_p}{\partial x_j^2}(\frac{B}{A}) = -\frac{2A^4}{B^2(B+A^2)} < 0$ we obtain an absolute maximum at $x_2 = \frac{B}{A}$, i.e. the absolute minimum of the pseudo-weight function over $[m, M]$ is at one of the endpoints of $[m, M]$. Switching x_j to either m or M generates now a vector with smaller pseudo-weight. The claim follows by repeating this procedure for all $m < x_l < M$. \square

We are ready now for the main result of this subsection.

THEOREM 3.7. *Let $\mathbf{x} = (x_1, \dots, x_n)$ be an arbitrary vector with non-negative integer components. Let $N \triangleq |\text{supp}(\mathbf{x})|$, and let M and m be the largest and smallest value among the non-zero entries of \mathbf{x} , respectively. Then*

$$w_p(\mathbf{x}) \geq N \cdot \frac{4mM}{(m+M)^2},$$

with equality if and only if \mathbf{x} contains $N \cdot \frac{M}{m+M}$ components of value m , and $N \cdot \frac{m}{m+M}$ components of value M .

PROOF. According to Lemma 3.6 we only consider vectors with t_m components equal to m and t_M components equal to M . With $t_M = N - t_m$, the pseudo-weight of \mathbf{x} is given as function f of t_m :

$$f(t_m) = w_p(\mathbf{x}) = \frac{(mt_m + Mt_M)^2}{m^2t_m + M^2t_M} = \frac{(mt_m + M(N - t_m))^2}{m^2t_m + M^2(N - t_m)}.$$

Its first and second derivative are then given by, respectively,

$$\begin{aligned} \frac{\partial f(t_m)}{\partial t_m} &= -\frac{((m-M)t_m + MN)((-m+M)(M-m)^2)t_m + MN(M-m)^2}{(m^2t_m + M^2N - M^2t_m)^2}, \\ \frac{\partial^2 f(t_m)}{\partial t_m^2} &= -\frac{2(mM(M-m)N)^2}{(m^2t_m + M^2N - M^2t_m)^3}. \end{aligned}$$

We see that the critical points of the function f are at $t_{m,1} = \frac{MN}{M-m}$ and $t_{m,2} = \frac{-MN(M-m)^2}{-(m+M)(M-m)^2} = \frac{MN}{m+M}$, with second derivative given by $\frac{\partial^2 f}{\partial t_m^2}(t_{m,1}) = -\frac{2(M-m)^2}{mMN} < 0$ and $\frac{\partial^2 f}{\partial t_m^2}(t_{m,2}) = \frac{2(M-m)^2}{mMN} > 0$, respectively. Hence the function f has an absolute minimum at $t_{m,2} = \frac{MN}{m+M}$ with value $f(t_{m,2}) = \frac{4mMN}{(m+M)^2}$. Finally, we calculate $t_M = N - t_{m,2} = \frac{mN}{m+M}$. \square

TABLE 1. Lower bounds on the pseudo-weight.

M	L	M	L	M	L
1	N	6	$\frac{24}{49}N$	11	$\frac{11}{36}N$
2	$\frac{8}{9}N$	7	$\frac{7}{16}N$	12	$\frac{48}{169}N$
3	$\frac{3}{4}N$	8	$\frac{32}{81}N$	13	$\frac{13}{49}N$
4	$\frac{16}{25}N$	9	$\frac{9}{25}N$	14	$\frac{56}{225}N$
5	$\frac{5}{9}N$	10	$\frac{40}{121}N$	15	$\frac{15}{64}N$

COROLLARY 3.8. Let $\mathbf{x} = (x_1, \dots, x_n)$ be an arbitrary vector with non-negative integer components. Let $N \triangleq |\text{supp}(\mathbf{x})|$, let M and m be the largest and smallest value among the non-zero entries of \mathbf{x} , respectively, and let $r \triangleq \frac{M}{m}$. Then

$$w_p(\mathbf{x}) \geq N \cdot \frac{4r}{(r+1)^2},$$

with equality if and only if \mathbf{x} contains $N \cdot \frac{r}{r+1}$ components of value m , and $N \cdot \frac{1}{r+1}$ components of value M .

PROOF. This follows directly from Theorem 3.7. □

It is easy to construct pseudo-codewords whose pseudo-weight achieves the lower bounds in Theorem 3.7 and Cor. 3.8, e.g. pseudo-codewords where all non-zero entries have the same value achieve these lower bounds.

EXAMPLE 3.9. For vectors of length N with $m = 1$ the corresponding lower bound L of the pseudo weight $w_p(\mathbf{x})$ is given in Table 1.

3.2. Bounds on the Pseudo-Weight of Pseudo-Codewords. We will now apply the general results obtained in the Section 3.1 to pseudo-codewords. In particular we will give a lower bound on the pseudo-weight of pseudo-codewords with entries equal to zero, one, and two only, as these are the ones suspected to have the smallest pseudo-weight among all *minimal* pseudo-codewords that are not codewords. We will also look at the pseudo-codewords of maximum support n , as these are candidates for having the largest pseudo-weight among all *minimal* pseudo-codewords that are not codewords. In the following, we will use the combinatorics terminology of a multiset $\{t_0 \cdot 0, t_1 \cdot 1, \dots, t_M \cdot M\}$ to display the values taken by the components together with the number of times a value $i \in \{0, 1, \dots, M\}$ is taken by the components of $\boldsymbol{\omega}$. Naturally, $|\text{supp}(\boldsymbol{\omega})| = \sum_{i=1}^M t_i$. We begin our considerations with a lemma.

LEMMA 3.10. Let \mathcal{C} be a code from projective plane over \mathbb{F}_q , with $H_{\text{PG}(2,q)}$ an $n \times n$ circulant parity-check matrix. Let $\boldsymbol{\omega}$ be a nonzero pseudo-codeword with the

set of entries given by the multiset $\{t_0 \cdot 0, t_1 \cdot 1, t_2 \cdot 2\}$. Then:

$$|\text{supp}(\boldsymbol{\omega})| \geq \frac{3}{2}(q+2).$$

PROOF. Applying Lemma 3.1 we obtain

$$\sum_{l=1}^2 l \cdot t_l \geq 2(q+2).$$

Since $\boldsymbol{\omega}$ reduced modulo 2 is a codeword, we have that $t_1 \geq q+2$. Adding the last 2 equations yields

$$2(t_1 + t_2) \geq 3(q+2) \quad \Rightarrow \quad 2|\text{supp}(\boldsymbol{\omega})| \geq 3(q+2).$$

The desired bound now follows. \square

COROLLARY 3.11. *Let \mathcal{C} be a code from projective plane over \mathbb{F}_q with $\mathbf{H}_{\text{PG}(2,q)}$ an $n \times n$ circulant parity-check matrix. Let $\boldsymbol{\omega}$ be a nonzero pseudo-codeword with the set of entries given by the multiset $\{t_0 \cdot 0, t_1 \cdot 1, t_2 \cdot 2\}$. Then the inequality holds*

$$w_p(\boldsymbol{\omega}) \geq \frac{4}{3}(q+2).$$

PROOF. Applying Lemma 3.10 together with Theorem 3.7 we obtain:

$$w_p(\boldsymbol{\omega}) \geq \frac{8}{9} \cdot \frac{3}{2}(q+2) = \frac{4}{3}(q+2).$$

\square

3.3. Maximization Of The Pseudo-Weight. In this section we give a way of constructing a minimal pseudo-codeword of largest possible support n , in a projective plane over \mathbb{F}_q , for any q even. We conjecture that this has the largest pseudo-weight among all minimal pseudo-codewords that are not codewords.

THEOREM 3.12. *Let \mathcal{C} be a code from projective plane over \mathbb{F}_q , with $\mathbf{H}_{\text{PG}(2,q)}$ an $n \times n$ circulant parity-check matrix. Let $L = \{P_j \mid j \in \{1, 2, \dots, q+1\}\}$ be an arbitrary line in the projective plane, and let \mathcal{S} be its support. Let $\boldsymbol{\omega} \in \mathbb{R}^n$ such that $\omega_i = q$ for all $i \in \mathcal{S}$ and $\omega_i = 1$ for all $i \in \{1, 2, \dots, n\} - \mathcal{S}$. Then $\boldsymbol{\omega}$ is a minimal pseudo-codeword with pseudo-weight equal to $\frac{(2q+1)^2}{q+2}$.*

REMARK 3.13. This vector is obtain by assigning values of q to the points on a line at infinity, and assigning values of 1 to the points of the remaining affine plane.

PROOF. Each point P_i of associated value $\omega_i = q$, belongs to q other lines, all intersecting L in exactly one point. Each of these q lines has a set of $q+1$ points of associated values q for the point P_i , and 1 for the remaining q points. Among the fundamental cone inequalities associated to the equation of line L , only one will be satisfied with equality, $-q+1+1+\dots+1=0$, all the other inequalities being satisfied strictly. This is valid for each point on the line L , hence there are a total number of $q(q-1) = q^2 - q = n - 1$ equations satisfied by $\boldsymbol{\omega}$. Thus far we obtained that $\boldsymbol{\omega}$ is a pseudo-codeword that attains $n-1$ equality among the defining inequalities of the fundamental cone. We need to show that the pseudo-codeword is minimal, i.e. we need to show that the matrix \mathbf{A} such that $\mathbf{A} \cdot \boldsymbol{\omega} = 0$ over the real field of numbers, has maximum possible rank $n-1$. Computing the matrix $\mathbf{A}\mathbf{A}^T$ results in the matrix

$$\mathbf{A}\mathbf{A}^T = q\mathbf{I}_{n-1} + \mathbf{1}_{n-1}$$

with 1_{n-1} the matrix with all entries equal to 1. The eigenvalues of the matrix AA^T are given by q , (with multiplicity $n-2$) and q^2+2q (with multiplicity 1).

It implies that the matrix AA^T is positive definite, hence, $A^T\omega = 0$ if and only if $\omega = 0$. It follows that the rank of A is equal to $n-1$. \square

THEOREM 3.14 (Conjecture). *If ω is a minimal pseudo-codeword that is not a codeword, then*

$$w_p(\omega) \leq \frac{(2q+1)^2}{q+2}.$$

4. Conclusions

In this paper we discussed some upper and lower bounds on the pseudo-weight of pseudo-codewords of type-I PG-LDPC codes, and gave examples of pseudo-codewords attaining the bounds. We also conjectured that the low support, low entry integer minimal pseudo-codewords are the ones of minimal pseudo-weight among the non-codeword minimal pseudo-codewords.

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