

Convolutional Codes with Maximum Distance Profile*

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Abstract

Maximum distance profile codes are characterized by the property that two trajectories which start at the same state and proceed to a different state will have the maximum possible distance from each other relative to any other convolutional code of the same rate and degree.

In this paper we use methods from systems theory to characterize maximum distance profile codes algebraically. The main result shows that maximum distance profile codes form a generic set inside the variety which parameterizes the set of convolutional codes of a fixed rate and a fixed degree.

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1 Introduction

The concept of maximum distance profile codes was introduced in [3]. This concept is closely related to the concept of optimum distance profile code widely studied in the convolutional code area; see e.g. [7, 8]. It was shown in [3] that maximum distance profile codes exist when the transmission rate is $(n - 1)/n$, and it was conjectured that such codes exist for every transmission rate. In systems theoretic terms, this means that the existence of maximum distance profile codes was established for multi-input, one-output systems and that it was conjectured for general multi-input, multi-output (MIMO) systems.

The main result of this paper shows the existence of maximum distance profile codes for general MIMO systems and that these codes are generic in the sense of algebraic geometry. The techniques we are using to establish this result are based on very classical results from linear systems theory. Thus, we will first explain the problem in terms of linear systems theory. For this, we follow the description as it can be found in [10, 12].

Let \mathbb{F} be a finite field. Let n, k , and δ be positive integers with $k < n$. Consider the matrices $A \in \mathbb{F}^{\delta \times \delta}$, $B \in \mathbb{F}^{\delta \times k}$, $C \in \mathbb{F}^{(n-k) \times \delta}$, and $D \in \mathbb{F}^{(n-k) \times k}$. A rate k/n convolutional code \mathcal{C} of degree δ can be described by the linear system governed by the equations:

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t, \\ y_t &= Cx_t + Du_t, \\ v_t &= \begin{pmatrix} y_t \\ u_t \end{pmatrix}, \quad x_0 = 0. \end{aligned} \tag{1.1}$$

We call $x_t \in \mathbb{F}^\delta$ the *state vector*, $u_t \in \mathbb{F}^k$ the *information vector*, $y_t \in \mathbb{F}^{n-k}$ the *parity vector*, and $v_t \in \mathbb{F}^n$ the *code vector*, each at time t . The set of all possible code vector sequences $v_t \in \mathbb{F}^n$ is called the convolutional code generated by (A, B, C, D) ; its elements are called codewords. For simplicity, we will assume that (A, B) forms a controllable pair and (A, C) forms an observable pair. We will refer to such a code as an (n, k, δ) -code.

By iterating the equations defining the system (1.1), it can be seen that a sequence $\{v_t = \begin{pmatrix} y_t \\ u_t \end{pmatrix} \in \mathbb{F}^n \mid t = 0, 1, 2, \dots, j\}$ represents the beginning of a codeword if and only if the following matrix equation is satisfied:

$$\left(\begin{array}{c|cccc} & D & & & \\ & CB & D & & \\ -I & CAB & CB & \cdots & \\ & \vdots & & \ddots & \ddots \\ & CA^{j-1}B & CA^{j-2}B & \cdots & CB & D \end{array} \right) \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_j \\ u_0 \\ u_1 \\ \vdots \\ u_j \end{pmatrix} = 0. \tag{1.2}$$

For the purpose of error control coding, it is important that any two codewords are far

apart with respect to a suitable metric. The following definition is fundamental in coding theory:

Definition 1.1 Let $x, y \in \mathbb{F}^n$ be vectors. The Hamming distance $\text{Ham}(x, y)$ is defined to be the number of components in which x and y differ. The weight $\text{wt}(x)$ of x is defined to be the number of nonzero components of x .

Clearly, one has that $\text{Ham}(x, y) = \text{wt}(x - y)$. When $\{v_t = \begin{pmatrix} y_t \\ u_t \end{pmatrix} \in \mathbb{F}^n \mid t = 0, 1, 2, \dots\}$ is a codeword, one defines its weight to be $\sum_t \text{wt}(v_t)$.

In this paper, we are concerned only with finite-weight codewords. These are defined as follows:

Definition 1.2 A sequence $\{v_t = \begin{pmatrix} y_t \\ u_t \end{pmatrix} \in \mathbb{F}^n \mid t = 0, 1, 2, \dots\}$ represents a *finite-weight codeword* if

1. Equation (1.1) is satisfied for all $t \in \mathbb{Z}_+$, where \mathbb{Z}_+ denotes the set of positive integers;
2. There is an integer j such that $x_{j+1} = 0$ and $u_t = 0$ for $t \geq j + 1$.

A well-studied concept in convolutional coding theory is that of column distances [8]. We give a systems theoretic definition.

Definition 1.3 The weight $\text{wt}(v)$ of a vector v is the number of nonzero components of v .

Definition 1.4 The j th column distance of the code \mathcal{C} is defined as

$$d_j := \min \left\{ \sum_{t=0}^j \text{wt}(u_t) + \sum_{t=0}^j \text{wt}(y_t) \right\},$$

where the minimum is taken over all trajectories (u_t, y_t) of the system (1.1) with initial vector $u_0 \neq 0$.

Clearly, one has that $d_0 \leq d_1 \leq d_2 \leq \dots$, and hence there exists an integer r such that $d_r = d_{r+j}$ for all $j \geq 0$. This largest possible column distance is of central importance in coding theory:

Definition 1.5

$$d_{\text{free}} := \lim_{j \rightarrow \infty} d_j \tag{1.3}$$

is called the *free distance* of the code \mathcal{C} .

Codes with a large free distance and the largest possible column distances are very desirable. The following two results give estimates for these parameters. The first one was proven in [3]:

Proposition 1.6 For every $j \in \mathbb{N}_0$, we have

$$d_j \leq (n - k)(j + 1) + 1.$$

The proof given in [3] uses algebraic properties of the parity check matrix. The following systems theoretic proof is almost trivial.

PROOF: Take an input sequence u_0, \dots, u_j , where $\text{wt}(u_0) = 1$ and $u_s = 0$ for $s \geq 1$. Let $y_0, \dots, y_j \in \mathbb{F}^{n-k}$ be the corresponding output sequence. Then,

$$d_j \leq \sum_{t=0}^j \text{wt}(u_t) + \sum_{t=0}^j \text{wt}(y_t) \leq (n - k)(j + 1) + 1.$$

□

The following theorem gives an upper bound for the free distance.

Theorem 1.7 ([11]) The free distance of an (n, k, δ) -code satisfies

$$d_{\text{free}} \leq (n - k) \left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1 \right) + \delta + 1. \quad (1.4)$$

The bound on the right hand side is called the *generalized Singleton bound*. With these preliminaries, we can now give the following definitions:

Definition 1.8 Let \mathcal{C} be an (n, k, δ) -code with column distances d_j and free distance d_{free} .

1. \mathcal{C} is said to have a *maximum distance profile* if

$$d_j = (n - k)(j + 1) + 1 \text{ for } j = 0, \dots, L := \left\lfloor \frac{\delta}{k} \right\rfloor + \left\lfloor \frac{\delta}{n - k} \right\rfloor.$$

2. \mathcal{C} is called an *MDS code* if d_{free} attains the generalized Singleton bound (1.4).
3. \mathcal{C} is called a *strongly MDS code* if

$$d_M = (n - k) \left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1 \right) + \delta + 1 \text{ for } M = \left\lfloor \frac{\delta}{k} \right\rfloor + \left\lceil \frac{\delta}{n - k} \right\rceil.$$

In [11, 13], it was shown that, for any rate k/n and degree δ , MDS codes form a generic set in the variety parametrizing convolutional codes of rate k/n and degree δ . In [3], the existence of $(n, n - 1, \delta)$ strongly MDS codes was established. When $n - k$ divides δ , we have $M = L$. It follows that, in this situation, a convolutional code has a maximum distance profile if and only if it is strongly MDS. In Theorem 2.10 of this paper, we establish the existence of maximum distance profile convolutional codes for all parameters (n, k, δ) over a suitably large base field \mathbb{F} .

The following Lemma gives an algebraic criterion for a minor to be trivially zero.

Lemma 2.3 *Let $\mathcal{T}_j \in \mathbb{F}^{(j+1)(n-k) \times (j+1)k}$ be a block Toeplitz matrix as defined above. Suppose that, for all integers i with $0 \leq i \leq j$, all entries of the matrix F_i are nonzero. Then, a minor $M_{j_1, j_2, \dots, j_r}^{i_1, i_2, \dots, i_r}$ of \mathcal{T}_j is not trivially zero if and only if its indices $i_1 < \dots < i_r \leq (j+1)(n-k)$ and $j_1 < \dots < j_r \leq (j+1)k$ satisfy*

$$j_t \leq \left\lceil \frac{i_t}{n-k} \right\rceil k \text{ for } t = 1, \dots, r. \quad (2.2)$$

PROOF: In this proof, we refer to both a submatrix of \mathcal{T}_j and its determinant as a minor. Let $M_{j_1, j_2, \dots, j_r}^{i_1, i_2, \dots, i_r}$ be a minor of \mathcal{T}_j . We first prove that this minor is trivially zero if and only if at least one of its diagonal entries is zero. We then prove that this minor has at least one zero on its diagonal if and only if there is a $t \in \{1, \dots, r\}$ such that

$$j_t > \left\lceil \frac{i_t}{n-k} \right\rceil k.$$

To prove the first equivalence, suppose first that the minor $M_{j_1, j_2, \dots, j_r}^{i_1, i_2, \dots, i_r}$ has at least one zero on its diagonal. We denote the entry in row m and column n of $M_{j_1, j_2, \dots, j_r}^{i_1, i_2, \dots, i_r}$ by (m, n) . Note that, if $(m, n) = 0$, then $(m', n') = 0$ for all entries (m', n') with $m' \leq m$ and $n' \geq n$. Let S_r denote the permutation group on r letters. From the expression

$$\sum_{\sigma \in S_r} (\text{sgn } \sigma) (1, \sigma(1)) (2, \sigma(2)) \dots (r, \sigma(r)) \quad (2.3)$$

giving the determinant of an $r \times r$ matrix over a commutative ring, we see that this minor is zero whether viewed as an element of $\mathbb{F}^{r \times r}$ or as an element of $R^{r \times r}$ in the manner defined above; in other words, it is trivially zero. We now prove the other direction. It is easy to see that every trivially zero 1×1 and 2×2 minor of \mathcal{T}_j has a zero on its diagonal. Let n be a positive integer with $n \geq 3$, and suppose that, for $r \leq n-1$, every trivially zero $r \times r$ minor has a zero on its diagonal. Suppose that the $n \times n$ minor $M_{j_1, j_2, \dots, j_n}^{i_1, i_2, \dots, i_n}$ is trivially zero. We use an induction argument to show that this minor has at least one zero on its diagonal. Notice that, when $M_{j_1, j_2, \dots, j_n}^{i_1, i_2, \dots, i_n}$ is viewed as an element of $R^{n \times n}$ in the manner defined above, the entry $(n, 1)$ appears exactly once. If this entry is zero, then every entry in the minor is zero, and thus all diagonal entries are zero. Suppose this entry is not zero. Doing a cofactor expansion along the first column shows that the $(n-1) \times (n-1)$ minor $M_{j_2, \dots, j_n}^{i_1, i_2, \dots, i_{n-1}}$ is trivially zero. By the induction hypothesis, this minor must have a zero on its diagonal. Thus, there is an entry $(s, s+1) = 0$ in $M_{j_1, j_2, \dots, j_n}^{i_1, i_2, \dots, i_n}$, which means that the minor $M_{j_{s+1}, \dots, j_n}^{i_1, i_2, \dots, i_s}$ is such that all of its entries are zero. Because we assumed $M_{j_1, j_2, \dots, j_n}^{i_1, i_2, \dots, i_n}$ is trivially zero, it follows that at least one of the minors $M_{j_1, j_2, \dots, j_s}^{i_1, i_2, \dots, i_s}$, $M_{j_{s+1}, \dots, j_n}^{i_{s+1}, \dots, i_n}$ is trivially zero. By the induction hypothesis, at least one of these minors has a zero on its diagonal. As the diagonals of these minors lie on the diagonal of $M_{j_1, j_2, \dots, j_n}^{i_1, i_2, \dots, i_n}$, we are done.

To prove the second equivalence, we simply note that the diagonal entries of this minor are the entries $(i_1, j_1), (i_2, j_2), \dots, (i_r, j_r)$ of \mathcal{T}_j . From the structure of \mathcal{T}_j , it is clear that the diagonal entry (i_t, j_t) is zero if and only if

$$j_t > \left\lceil \frac{i_t}{n-k} \right\rceil k.$$

□

Theorem 2.4 *Let \mathcal{C} be an (n, k, δ) convolutional code described by matrices (A, B, C, D) and consider the block Toeplitz matrix \mathcal{T}_j introduced in (2.1). Then \mathcal{C} has j th column distance $d_j = (n - k)(j + 1) + 1$ if and only if every minor which is not trivially zero is nonzero.*

PROOF: \Leftarrow : Suppose that

$$\left(\begin{array}{cccc|cccc} y_0 & y_1 & \dots & y_j & u_0 & u_1 & \dots & u_j \end{array} \right)^T$$

is a finite-weight codeword with $u_0 \neq 0$ and that the vector

$$\left(\begin{array}{cccc} u_0 & u_1 & \dots & u_j \end{array} \right)^T$$

has weight r . Suppose that every minor of the matrix (2.1) which is not trivially zero is nonzero. Because $u_0 \neq 0$, this means that at most $r - 1$ rows of (2.1) are in the left kernel of

$$\left(\begin{array}{cccc} u_0 & u_1 & \dots & u_j \end{array} \right)^T.$$

Thus, this codeword has weight at least $r + (j + 1)(n - k) - (r - 1) = (j + 1)(n - k) + 1$. We therefore have that the weight of any codeword with $u_0 \neq 0$ is at least $(j + 1)(n - k) + 1$. In other words, $d_j \geq (j + 1)(n - k) + 1$. Proposition 1.6 implies that $d_j = (j + 1)(n - k) + 1$.

\Rightarrow : We prove the contrapositive. We first note that the result follows trivially if, for some integer i with $0 \leq i \leq j$, the matrix F_i contains a zero entry. We therefore assume that all such entries are nonzero. Suppose that the matrix (2.1) has an $r \times r$ minor which is zero but not trivially zero, where $r \geq 2$ (in this proof, we again use 'r × r minor' to refer to both the submatrix and its determinant). If the r rows of this minor belong to the left kernel of a column vector of weight r , then, because of the structure of (2.1), we can form a nonzero vector

$$\left(\begin{array}{cccc} u_0 & u_1 & \dots & u_j \end{array} \right)^T$$

of weight r with $u_0 \neq 0$ such that the weight of

$$\left(\begin{array}{cccc|cccc} y_0 & y_1 & \dots & y_j & u_0 & u_1 & \dots & u_j \end{array} \right)^T$$

is at most $r + (j + 1)(n - k) - r = (j + 1)(n - k)$. If not, then the r rows of this minor belong to the kernel of a nonzero column vector of weight $r' \leq r$. The r' nonzero components of this vector pick out an $r \times r'$ submatrix. We would like to see that this submatrix contains

an $r' \times r'$ minor which is zero but not trivially zero. To obtain this minor, we simply choose the bottom r' rows. This minor is clearly zero. Because we have assumed that the entries of the matrices F_i are all nonzero, we know from Lemma 2.3 that a minor is not trivially zero if and only if its i th column has the property that the last $r + 1 - i$ entries are nonzero. The columns of the original $r \times r$ minor have this property. Thus, the columns of our $r' \times r'$ subminor have this property. Because of the structure of (2.1), we can form a nonzero vector

$$\begin{pmatrix} u_0 & u_1 & \dots & u_j \end{pmatrix}^T$$

of weight r' with $u_0 \neq 0$ such that the weight of

$$\begin{pmatrix} y_0 & y_1 & \dots & y_j & | & u_0 & u_1 & \dots & u_j \end{pmatrix}^T$$

is at most $r' + (j + 1)(n - k) - r' = (j + 1)(n - k)$. \square

Specializing Theorem 2.4 for $j = L$, and recalling [3, Corollary 2.4], we immediately get an algebraic criterion for an (A, B, C, D) convolutional code to represent a maximum distance profile code:

Corollary 2.5 *Let $L = \left\lfloor \frac{\delta}{k} \right\rfloor + \left\lfloor \frac{\delta}{n-k} \right\rfloor$. Then, the matrices (A, B, C, D) generate a maximum distance profile (n, k, δ) convolutional code if and only if the matrix \mathcal{T}_L has the property that every minor which is not trivially zero is nonzero.*

Remark 2.6 Theorem 2.4 and Corollary 2.5 give polynomial conditions in the entries of the matrices (A, B, C, D) which guarantee that a convolutional code has the maximum distance property. These algebraic properties are invariant under state space transformations. This means that, if (A, B, C, D) has this property, then so does $(SAS^{-1}, SB, CS^{-1}, D)$ for every matrix $S \in Gl_\delta(\mathbb{F})$. As a result, these conditions are really algebraic conditions on the quasi-projective variety [6, 9] describing the set of rate k/n convolutional codes of degree δ . We have therefore established that the set of convolutional codes having the maximum distance property form a Zariski open set in this quasi-projective variety.

The remainder of this section is devoted to showing that this Zariski open set of maximal distance profile codes is nonempty as soon as the base field is sufficiently large. As a first step toward this result, we have the following theorem.

Theorem 2.7 *Let j, k, n be fixed positive integers and consider the matrix introduced in (2.1). If the field \mathbb{F} is sufficiently large, then one can find a sequence of matrices $\{F_0, \dots, F_j\}$, where $F_i \in \mathbb{F}^{(n-k) \times k} \forall i \in \{0, 1, \dots, j\}$, such that every minor of the matrix \mathcal{T}_j which is not trivially zero is nonzero.*

PROOF: Let \mathbb{F} be an arbitrary finite field and let $\bar{\mathbb{F}}$ denote the algebraic closure of \mathbb{F} . Note that $\bar{\mathbb{F}}$ is an infinite field. To say that a minor of \mathcal{T}_j is zero but not trivially zero is to say that the entries of \mathcal{T}_j satisfy a nonzero polynomial equation in $\bar{\mathbb{F}}[x_1, x_2, \dots, x_{(j+1)(n-k)k}]$. As there are finitely many minors, there are finitely many such polynomial equations describing

those matrix sequences $\{F_0, \dots, F_j\}$ for which \mathcal{T}_j has at least one minor that vanishes but does not trivially vanish. Each of these polynomials describes a proper algebraic subset of $\overline{\mathbb{F}}^{(j+1)(n-k)k}$, the complement of which is a nonempty Zariski open set in $\overline{\mathbb{F}}^{(j+1)(n-k)k}$. We may take the intersection of these Zariski open sets, and, as there are finitely many of them, the result is again a nonempty Zariski open set in $\overline{\mathbb{F}}^{(j+1)(n-k)k}$. Take $\{F_0, \dots, F_j\}$ to be an element in this intersection. There are finitely many entries in this matrix sequence. Thus, either all of the entries belong to \mathbb{F} , or there is a finite extension field of \mathbb{F} containing all of them. In this case, we may instead take \mathbb{F} to be the smallest such extension. \square

In order to show the existence of maximum distance profile convolutional codes of arbitrary rate k/n , we use a result by Tether [14] from *minimal partial realization theory*. For a given rate, this result allows us to show the existence of such codes possessing only certain degrees. It will require only a small amount of additional work to obtain the existence of such codes of arbitrary degree. Readers interested in the minimal partial realization problem and its connections are referred to [1].

Theorem 2.8 *Let j, k , and n be fixed positive integers with $k < n$. Let $\{F_0, F_1, \dots, F_j\}$ and \mathcal{T}_j be as in Theorem 2.7. Let δ be the smallest positive integer such that*

$$\left\lfloor \frac{\delta}{k} \right\rfloor + \left\lfloor \frac{\delta}{n-k} \right\rfloor \geq j.$$

Suppose that $\{F_0, F_1, \dots, F_j\}$ is such that the corresponding \mathcal{T}_j has the property that its minors which are not trivially zero are nonzero. Then, one can extend the sequence $\{F_0, F_1, \dots, F_j\}$ to an infinite sequence $\{F_0, F_1, \dots\}$ of $(n-k) \times k$ matrices over \mathbb{F} such that the infinite block Hankel matrix

$$\mathcal{F} = \begin{pmatrix} F_1 & F_2 & F_3 & \cdots \\ F_2 & F_3 & F_4 & \cdots \\ F_3 & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

has rank δ .

PROOF: Let $\mathcal{F}_{x,y}$ denote the block Hankel matrix

$$\mathcal{F}_{x,y} = \begin{pmatrix} F_1 & F_2 & \cdots & F_y \\ F_2 & F_3 & \cdots & F_{y+1} \\ \vdots & \vdots & \cdots & \vdots \\ F_x & F_{x+1} & \cdots & F_{x+y-1} \end{pmatrix}$$

As was shown in [14, Theorem 1], any matrix sequence $\{F_1, \dots, F_j\}$ has a minimal partial realization of degree d , where

$$d = \sum_{i=1}^j \text{rank } \mathcal{F}_{i,j+1-i} - \sum_{i=1}^{j-1} \text{rank } \mathcal{F}_{i,j-i}.$$

It is easy to see that, up to a reordering of block columns, each $\mathcal{F}_{i,j+1-i}$ appearing in the first summation in the above formula for d is a submatrix of \mathcal{T}_j (take the intersection of the last i block rows and the first $j+1-i$ block columns). As each $\mathcal{F}_{i,j-i}$ in the second summation is a submatrix of $\mathcal{F}_{i,j+1-i}$, the same is true for these. By assumption, the only minors of \mathcal{T}_j that are zero are those that are trivially zero. Thus, $\text{rank } \mathcal{F}_{x,y} = \min((n-k)x, ky)$ for each $\mathcal{F}_{x,y}$ appearing in the above formula for d .

First, suppose that $k \geq n-k$. The formula for d then becomes

$$d = \sum_{i=1}^j \min(i(n-k), (j+1-i)k) - \sum_{i=1}^{j-1} \min(i(n-k), (j-i)k).$$

Suppose there exists an integer $r, 1 \leq r \leq j-1$, with $r(n-k) \geq (j-r)k$. Let i^* be the smallest such integer. Then, the formula for d becomes

$$d = (i^* - 1)(n - k) + (\min(i^*(n - k), (j - i^* + 1)k) - \min((i^* - 1)(n - k), (j - i^* + 1)k)).$$

By definition of i^* , $(i^* - 1)(n - k) < (j - i^* + 1)k$, so that the last term in this expression is $-(i^* - 1)(n - k)$. Thus,

$$d = \min(i^*(n - k), (j - i^* + 1)k).$$

Note that $i^* = \lceil j \frac{k}{n} \rceil$. Consider the difference

$$\begin{aligned} (j - i^* + 1)k - (i^* - 1)(n - k) &= (j - \lceil j \frac{k}{n} \rceil + 1)k - (\lceil j \frac{k}{n} \rceil - 1)(n - k) = \\ &= (j + 1)k - \lceil j \frac{k}{n} \rceil n + (n - k). \end{aligned}$$

We want to see that this number is at least $n - k$, or, equivalently, that $(j + 1)k - \lceil j \frac{k}{n} \rceil n \geq 0$. This will imply that $\min(i^*(n - k), (j - i^* + 1)k) = i^*(n - k)$. We have

$$(j + 1)k - \lceil j \frac{k}{n} \rceil n \geq 0 \iff (j + 1)k \geq \lceil j \frac{k}{n} \rceil n \iff \frac{(j + 1)k}{n} \geq \lceil j \frac{k}{n} \rceil.$$

By assumption, $k \geq n - k$, which means $k \geq \frac{n}{2}$. It follows that $\frac{(j+1)k}{n} \geq \lceil j \frac{k}{n} \rceil$. Thus, we have $d = i^*(n - k)$.

By assumption, $i^*(n - k) \geq (j - i^*)k$. We have

$$\begin{aligned} i^*(n - k) \geq (j - i^*)k &\iff i^*(n - k) + i^*k \geq jk \iff \frac{i^*(n - k)}{k} + i^* \geq j \\ &\iff \left\lfloor \frac{i^*(n - k)}{k} + i^* \right\rfloor = \left\lfloor \frac{i^*(n - k)}{k} \right\rfloor + \lfloor i^* \rfloor = \left\lfloor \frac{i^*(n - k)}{k} \right\rfloor + \left\lfloor \frac{i^*(n - k)}{n - k} \right\rfloor \geq j. \end{aligned}$$

Let $\delta = i^*(n - k)$. We want to see that δ is the smallest positive integer satisfying $\lfloor \frac{\delta}{k} \rfloor + \lfloor \frac{\delta}{n - k} \rfloor \geq j$. Consider $\lfloor \frac{\delta}{k} \rfloor = \lfloor \frac{i^*(n - k)}{k} \rfloor$. Since $i^* = \lceil j \frac{k}{n} \rceil$, we may write $i^* = \frac{jk + s}{n}$ where $s \in \mathbb{N}_0$ and $0 \leq s \leq n - 1$. Then, $\lfloor \frac{\delta}{k} \rfloor = \frac{znk - y}{nk}$ where $z, y \in \mathbb{N}_0$ and $0 \leq y \leq nk - 1$. Thus, we have

$$\left\lfloor \frac{\delta}{k} \right\rfloor + \left\lfloor \frac{\delta}{n - k} \right\rfloor = \frac{znk - y}{nk} + \frac{jk + s}{n} \leq \frac{(jk + s)(n - k)}{nk} + \frac{jk + s}{n}$$

with equality precisely when $y = 0$ or, in other words, when $k \mid i^*(n - k)$. Suppose $y = 0$. Then, since $s \leq n - 1$ and $k \geq \frac{n}{2}$, we have

$$\lfloor \frac{\delta}{k} \rfloor + \lfloor \frac{\delta}{n-k} \rfloor = \frac{(jk + s)(n - k)}{nk} + \frac{jk + s}{n} = \frac{jk + s}{k} = j + \frac{s}{k} < j + 2$$

As $j + \frac{s}{k}$ must be an integer, we may write $j + \frac{s}{k} \leq j + 1$. For the same reason, we must have $s = lk, l \in \{0, 1\}$. From this, we see that $\lfloor \frac{\delta}{k} \rfloor = \lfloor \frac{i^*(n-k)}{k} \rfloor$ is bounded above by $j + 1$, and this upper bound is obtained precisely when $k \mid i^*(n - k)$ and $l = 1$. Because $(n - k) \mid \delta$, it follows that when $l = 1$, $\lfloor \frac{\delta-1}{k} \rfloor + \lfloor \frac{\delta-1}{n-k} \rfloor \leq j - 1$, and when $l = 0$, that $\lfloor \frac{\delta-1}{k} \rfloor + \lfloor \frac{\delta-1}{n-k} \rfloor \leq j - 2$. If y is nonzero, then $\lfloor \frac{\delta}{k} \rfloor + \lfloor \frac{\delta}{n-k} \rfloor \leq j$, and so $\lfloor \frac{\delta-1}{k} \rfloor + \lfloor \frac{\delta-1}{n-k} \rfloor \leq j - 1$.

If no such r exists, there are two possibilities. The first is that $j(n - k) > k$, so that $\min(j(n - k), k) = k$. Then $d = (j - 1)(n - k) + k - (j - 1)(n - k) = k$. Because $(j - 1)(n - k) < k$, we have the inequalities $\frac{n}{n-k} = \frac{k}{n-k} + 1 > j > \frac{k}{n-k}$, and so we may write $\lfloor \frac{k}{n-k} \rfloor + 1 \geq j > \lfloor \frac{k}{n-k} \rfloor$. Letting $\delta = k$, we see that δ is the smallest positive integer satisfying $\lfloor \frac{\delta}{k} \rfloor + \lfloor \frac{\delta}{n-k} \rfloor \geq j$. The second possibility is that $j(n - k) \leq k$, so that $\min(j(n - k), k) = j(n - k)$. Let $\delta = j(n - k)$. Clearly, δ is the smallest positive integer satisfying $\lfloor \frac{\delta}{k} \rfloor + \lfloor \frac{\delta}{n-k} \rfloor \geq j$.

The proof for the case $k < n - k$ is similar. \square

Theorem 2.8 immediately establishes the existence of maximum distance profile codes for certain parameters (n, k, δ) :

Lemma 2.9 *Let k, n and δ be positive integers such that $k < n$ and either $k \mid \delta$ or $n - k \mid \delta$. Then, an (n, k, δ) maximum distance profile convolutional code exists over a sufficiently large base field.*

PROOF: Let $\lfloor \frac{\delta}{k} \rfloor + \lfloor \frac{\delta}{n-k} \rfloor = L$. Note that if δ decreases, then L must decrease. From the proof of Theorem 2.7, we know that there exists a finite extension field of \mathbb{F} and a sequence $\{F_0, F_1, \dots, F_L\}$ of $(n - k) \times k$ matrices with entries in this extension field such that the only minors of the corresponding \mathcal{T}_L which are zero are those which are trivially zero. From the proof of Theorem 2.8, we know that there exist matrices $A \in \mathbb{F}^{\delta \times \delta}$, $B \in \mathbb{F}^{\delta \times k}$, and $C \in \mathbb{F}^{(n-k) \times \delta}$ which give a minimal partial realization of the sequence $\{F_1, \dots, F_L\}$. Let $D = F_0$. Then, the matrices A, B, C , and D describe an (n, k, δ) convolutional code via (1.1). By Corollary 2.5, this code has a maximum distance profile. \square

We now state and prove the main Theorem.

Theorem 2.10 *Let k and n be positive integers such that $k < n$. Let δ be a positive integer. Then, there exists a maximum distance profile (n, k, δ) convolutional code over some finite extension field of \mathbb{F} . Moreover, the set of maximum distance profile (n, k, δ) convolutional codes forms a generic set in $\overline{\mathbb{F}}^{(\delta+n-k)(\delta-k)}$.*

PROOF: Let $L = \lfloor \frac{\delta}{k} \rfloor + \lfloor \frac{\delta}{n-k} \rfloor$. Let δ^* be the smallest integer satisfying this equality. If $L = 0$, the theorem is easily seen to be true. This is because the problem of proving existence is reduced to finding an (A, B, C, D) representation of an (n, k, δ) convolutional code over a finite extension field of \mathbb{F} such that $[-I \ D]$ represents the parity check matrix of an MDS block code. The proof that such a D can be found is essentially identical to the proof of Lemma 2.7. The set of such (A, B, C, D) representations is obviously a Zariski open set in $\overline{\mathbb{F}}^{(\delta+n-k)(\delta-k)}$. This proves that the set of such codes forms a generic set in $\overline{\mathbb{F}}^{(\delta+n-k)(\delta-k)}$. Thus, we may assume $L \neq 0$.

Let S denote the set of 4-tuples of matrices (A, B, C, D) with $A \in \mathbb{F}^{\delta \times \delta}$, $B \in \mathbb{F}^{\delta \times k}$, $C \in \mathbb{F}^{(n-k) \times \delta}$, and $D \in \mathbb{F}^{(n-k) \times k}$ and having the property that every minor in (2.1) which is not trivially zero is nonzero. We first show S is a nonempty Zariski open set in $\overline{\mathbb{F}}^{(\delta+n-k)(\delta-k)}$. S is obviously Zariski open. We now show it is nonempty. Let $r = \delta - \delta^*$. Let $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ be a representation of a maximum distance profile (n, k, δ^*) convolutional code; the existence of such a code is implied by Lemma 2.9. Consider the matrices

$$A = \left(\begin{array}{c|c} 0_{r \times r} & 0_{r \times \delta^*} \\ \hline 0_{\delta^* \times r} & \tilde{A} \end{array} \right), \quad B = \left(\begin{array}{c} 0_{r \times k} \\ \hline \tilde{B} \end{array} \right), \quad C = (0_{(n-k) \times r} \mid \tilde{C}), \quad D = \tilde{D}.$$

Notice that $CA^{i-1}B = \tilde{C}\tilde{A}^{i-1}\tilde{B}$ for all $i \geq 1$. Because $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is a representation of a maximum distance profile convolutional code, we have shown that S is nonempty.

To complete the proof, we note that the reasoning used in the proof of Lemma 2.7 implies that 4-tuples of matrices (A, B, C, D) in $\overline{\mathbb{F}}^{(\delta+n-k)(\delta-k)}$ such that (A, B) is a controllable pair and (A, C) is an observable pair form a nonempty Zariski open set in $\overline{\mathbb{F}}^{(\delta+n-k)(\delta-k)}$. Intersecting this set with S gives a nonempty Zariski open set in $\overline{\mathbb{F}}^{(\delta+n-k)(\delta-k)}$ consisting of maximum distance profile (n, k, δ) convolutional codes. Thus, these codes form a generic set in $\overline{\mathbb{F}}^{(\delta+n-k)(\delta-k)}$. \square

It was pointed out above that, when $n - k \mid \delta$, an (n, k, δ) convolutional code has a maximum distance profile if and only if it is strongly MDS. With this, we have the following Corollary:

Corollary 2.11 *When $n - k \mid \delta$, there exists an (n, k, δ) strongly MDS convolutional code over a finite extension field of \mathbb{F} .*

3 Codes with Maximum Distance Profile in Terms of Polynomial Generator Matrices

In the coding literature, convolutional codes are usually studied via (polynomial) generator and parity check matrices. The relevant results presented in [3] were formulated in terms of such polynomial matrix descriptions. In this section, we make the connection between polynomial and state space descriptions of convolutional codes. For this, we follow [10, 12], where further details may be found as well. We also state a necessary and sufficient condition for a polynomial parity check matrix to define a maximum distance profile convolutional code.

Consider the transfer function $T(z) := C(zI - A)^{-1}B + D$. Let $P(z)^{-1}Q(z) = T(z)$ be a left coprime factorization of $T(z)$ and $H(z) := [P(z) \ Q(z)]$. Consider the polynomial vectors:

$$u(z) = u_0z^\gamma + u_1z^{\gamma-1} + \dots + u_\gamma; \quad u_t \in \mathbb{F}^k, t = 0, \dots, \gamma,$$

and

$$y(z) = y_0z^\gamma + y_1z^{\gamma-1} + \dots + y_\gamma; \quad y_t \in \mathbb{F}^{n-k}, t = 0, \dots, \gamma.$$

Then, the following conditions are equivalent.

1. The vectors u_t and y_t satisfy the state space equation (1.1).
2. The vectors u_t and y_t satisfy

$$\left(\begin{array}{c|cccccc} & D & & & & & \\ & CB & D & & & & \\ -I & CAB & CB & \ddots & & & \\ & \vdots & & \ddots & \ddots & & \\ & CA^{\gamma-1}B & CA^{\gamma-2}B & \dots & CB & D & \end{array} \right) \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\gamma \\ u_0 \\ u_1 \\ \vdots \\ u_\gamma \end{pmatrix} = 0. \quad (3.4)$$

3. There exists a ‘state vector’

$$x(z) = x_0z^\gamma + x_1z^{\gamma-1} + \dots + x_\gamma; \quad x_t \in \mathbb{F}^\delta, t = 0, \dots, \gamma,$$

such that

$$\begin{bmatrix} zI - A & 0_{\delta \times (n-k)} & -B \\ -C & I_{n-k} & -D \end{bmatrix} \begin{bmatrix} x(z) \\ y(z) \\ u(z) \end{bmatrix} = 0. \quad (3.5)$$

4. $\begin{bmatrix} y(z) \\ u(z) \end{bmatrix}$ is a code word, i.e.

$$H(z) \begin{bmatrix} y(z) \\ u(z) \end{bmatrix} = [P(z) \ Q(z)] \begin{bmatrix} y(z) \\ u(z) \end{bmatrix} = 0.$$

5. $y(z) = T(z)u(z)$

The proof of these equivalences is straightforward, and more details can be found, e.g., in [12].

We close with the following theorem.

Theorem 3.1 *Let $H(z) = \sum_{l=0}^{\mu} H_l z^l$ be the parity check matrix of an (n, k, δ) -code. Assume $H_l = 0$ for $l > \mu$. Let*

$$\mathcal{H}_j := \begin{pmatrix} H_0 & & & & \\ H_1 & H_0 & & & \\ \vdots & \vdots & \ddots & & \\ H_j & H_{j-1} & \cdots & H_0 & \end{pmatrix} \in \mathbb{F}^{(j+1)(n-k) \times (j+1)n},$$

Then $H(z)$ represents a code whose j th column distance $d_j = (n - k)(j + 1) + 1$ if and only if every $(j + 1)(n - k) \times (j + 1)(n - k)$ full-size minor formed from the columns with indices $1 \leq i_1 < \dots < i_{(j+1)(n-k)}$, where $i_{s(n-k)} \leq sn$ for $s = 1, \dots, j$, is nonzero.

In particular when

$$j := L := \left\lfloor \frac{\delta}{k} \right\rfloor + \left\lfloor \frac{\delta}{n - k} \right\rfloor.$$

then $H(z)$ represents a maximum distance profile code if and only if every $(L + 1)(n - k) \times (L + 1)(n - k)$ full-size minor formed from the columns with indices $1 \leq i_1 < \dots < i_{(L+1)(n-k)}$, where $i_{s(n-k)} \leq sn$ for $s = 1, \dots, L$, is nonzero.

PROOF: This theorem is a direct consequence of [3, Corollary 2.4] and [3, Theorem 5.3]. \square

4 Conclusion

In this paper, we established the existence of maximum distance profile codes for all transmission rates and all degrees. The main results are existence results. Important questions remain open as to how maximum distance profile codes may be constructed and the minimal field size required for doing so. For the construction of such codes we found that many cyclic convolutional codes [4] have the maximum distance profile property and this might be a promising avenue for constructing such codes.

The properties of these codes are very appealing for error control coding; the distance between two trajectories which start at a common initial state is maximal, and hence these codes have the potential to have the maximal amount of errors per time interval corrected. In applications where fault-diagnosis is important (see e.g. [2, 5]), it has already been pointed out that codes with maximal free distance and hence also codes with a maximal distance profile are very important.

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