Title: Understanding the Paris-Harrington Theorem, Part I

Abstract:
Call a finite set of natural numbers $X$ relatively large if $|X| \geq \min(X)$. Given natural numbers $M$, $e$, $r$, and $k$, let $M \rightarrow^e_r (k)$ denote the statement that for every partition $P : [M]^e \rightarrow r$ of the size $e$ subsets of $M$ into $r$ colors, there is a relatively large set $X \subseteq M$ that is homogeneous for $P$. It is not difficult to show, using the infinite Ramsey theorem and a compactness argument, that

\[ (*) \quad (\forall e, r, k)(\exists M)(M \rightarrow^e_r (k)). \]

However, Paris and Harrington\(^1\) showed that the statement (*) cannot be proved in Peano arithmetic. We will try to present a clear explanation of their proof.

\(^1\)A Mathematical Incompleteness in Peano Arithmetic, Handbook of Mathematical Logic, 1977