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Math 80220: Topics in Algebra. Coxeter groups and Hecke algebras

The finite Coxeter groups are the finite groups generated by reflections on real Euclidean spaces. Examples include dihedral groups, the symmetry groups of regular polytopes (e.g. regular polygons and platonic solids) and the Weyl groups of semisimple complex Lie groups and Lie algebras (such as the special linear group and Lie algebra). General Coxeter groups may be defined as certain (special) amalgamated products of dihedral groups or realized as discrete reflection groups on real vector spaces (or more general manifolds or cell complexes). They, and particularly the crystallographic Coxeter groups, have a very rich theory, with interconnections to many different areas of mathematics, and significant applications in several of them, and may be studied by a variety of algebraic, geometric and combinatorial techniques.

One of their important areas of applications is by means of their associated Hecke algebras. These are deformations of group algebras of the Coxeter group which arose historically as endomorphism algebras of certain induced representation, and they now play an important role in many areas of representation theory (e.g. of finite simple groups of Lie type, and p-adic groups). Fundamental progress in many of these areas became possible following the definition by Kazhdan and Lusztig in 1979 of a new basis, with great representation-theoretic and geometric significance in some cases, of the generic Hecke algebra. For example, the Kazhdan-Lusztig conjecture determined key aspects of the structure of categories of infinite-dimensional highest weight representations of semisimple complex Lie algebras in terms of this basis for the Hecke algebra of the corresponding finite Weyl group, and the conjecture was proved using deep methods involving D-modules, perverse sheaves and machinery from the proof of the Weil conjectures. Subsequently, the proof was refined by Soergel to a construction, involving a categorification of the Hecke algebra, of the relevant highest weight categories directly from the reflection representation of the Weyl group. The construction extends to general Coxeter groups modulo an elementary conjecture which was proved in 2013 by Elias and Williamson using ideas from Hodge theory. This provides an elementary proof of the Kazhdan-Lusztig conjecture and implies that various categories of fundamental representation-theoretic and geometric significance are part of natural families which extend beyond the traditional association of Lie theory with crystallographic Coxeter groups.

This course will give an account of some of these developments, keeping prerequisites to a minimum by focusing mostly on the parts of the ideas directly involving Coxeter groups. The first part will give an introduction to Coxeter groups, root systems, Bruhat order and Hecke algebras, from mostly algebraic and combinatorial points of view. The second part will discuss categorification of the Hecke algebra and the structure of highest weight categories associated to Coxeter groups, and will require more generalities from commutative and homological algebra, category theory, representation theory etc; necessary background and motivation will be provided but it will not always be possible to supply full details. Some knowledge of Lie theory would be helpful for motivation but will not be logically essential for most of the material covered. The precise balance between the two parts of the course will depend to some extent on the interests and background of those attending.