Title: Computability strength of $\mathbb{R}$ compared to $2^\omega$

Abstract:
In recent work, Noah Schweber defines a reducibility notion for structures $A$ and $B$, potentially uncountable. The idea of the reducibility is that $A \leq^*_\omega B$ if, after a forcing collapse that causes $A$ and $B$ to become countable, every copy of $B$ computes a copy of $A$. This notion turns out to be independent of the specific forcing that is used.

From this point of view, we wish to consider two structures: the field $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$, and the two-sorted power set of $\omega$, $\mathcal{W} = (\mathcal{P}(\omega) \cup \omega, \in, S)$. After forcing collapse, $\mathcal{R}$ continues to be an Archimedean real closed field, and $\omega$ continues to be $\omega$, so $\mathcal{W}$ continues to be a Scott set.

Schweber proved that $\mathcal{W} \leq^*_\omega \mathcal{R}$. (The binary expansions of the elements of $\mathcal{R}$ are precisely the subsets of $\omega$ that can be found in $\mathcal{W}$, before and after forcing collapse.) We prove that the converse does not hold: in any forcing extension of $V$ that collapses $2^\omega$, there is a copy of $\mathcal{W}$ that cannot compute a copy of $\mathcal{R}$.

This proves that the computability theorist’s $\mathbb{R}$ is different from the analyst’s $\mathbb{R}$ in a fairly intrinsic manner: the various problems that arise with diadic rationals, or with equality of Cauchy sequences are unavoidable when translating between $2^\omega$ and $\mathbb{R}$. 