Graduate Basic Courses

Algebra I, II – 60210, 60220

The examinable material for the graduate algebra candidacy exam is 1 through the first part of 3 below (up to but not including categories), though Algebra I will usually cover more than this. Topics labeled *, and perhaps additional topics not mentioned, may be covered at the discretion of the instructor.

1. Groups
Groups. Cyclic groups, permutation groups symmetric and alternating groups, matrix groups. Subgroups, quotient groups, direct products, homomorphism theorems. Automorphisms, conjugacy. Cosets, Lagrange's theorem. Group actions, G-sets, Sylow theorems, free groups and presentations.

2. Rings
Rings. Polynomial rings, matrix rings. Ideals, quotient rings, homomorphism theorems. Prime and maximal ideals. Principal ideal domains, Euclidean domains, unique factorization domains. Localization of rings, field of fractions.

3. Modules
Finite dimensional vector spaces. Basis, dimension, linear transformations, matrices. Modules. Submodules, quotient modules, homomorphism theorems, structure of finitely generated modules over a principal ideal domain. Language of categories and functors. Direct sums and products, free, projective and injective modules. Duality, multilinear forms, determinants.

4. Canonical forms of matrices of linear transformations
Jordan, rational and primary rational canonical forms. Invariant factors and elementary divisors.

5. Fields
Fields, algebraic and transcendental field extensions, degree, transcendence degree, algebraic closure. Fundamental theorem of Galois theory, separability, normality. Finite fields. *Cyclotomic extensions, *cyclic extensions, *solvable and nilpotent groups, *Impossibility proofs (trisecting angles, etc), *solvability of polynomial equations by radicals

6. Tensor products
Tensor products, *algebras, *(the tensor, symmetric and exterior algebras)

7. Chain conditions
Artinian and Noetherian rings and modules, Hilbert basis theorem. *Simple and semisimple modules.

8. Commutative algebra
Localization of modules. Integral ring extensions. *Localization and spec of a ring. *Nakayama's lemma, *Noether normalization lemma. *Nullstellensatz. *Affine algebraic varieties and commutative rings.

References

T. Hungerford, *Algebra, Graduate Texts in Mathematics 73.*
N. Jacobson, *Basic algebra I, II*. S. Lang, *Algebra*.

Real Analysis I, II – 60350, 60360

1. Calculus
Calculus of one and several variables, the Implicit and Inverse Function Theorems, pointwise and uniform convergence of sequences of functions, integration and differentiation of sequences, the Weierstrass Approximation Theorem.

2. Lebesgue measure and integration on the real line
Measurable sets, Lebesgue measure, measurable functions, the Lebesgue integral and its relation to the Riemann integral, convergence theorems, functions of bounded variation, absolute continuity and differentiation of integrals.

3. General measure and integration theory
Measure spaces, measurable functions, integration convergence theorems, signed measures, the Radon-Nikodym Theorem, product measures, Fubini’s Theorem, Tonelli’s Theorem.

4. Families of functions
Equicontinuous families and the Arzela-Ascoli Theorem, the Stone-Weierstrass Theorem.

5. Banach spaces
L_P- spaces and their conjugates, the Riesz-Fisher Theorem, the Riesz Representation Theorem for bounded linear functionals on L_P, C(X), the Riesz Representation Theorem for C(X), the Hahn-Banach Theorem, the Closed Graph and Open Mapping Theorems, the Principle of Uniform Boundedness, Alaoglu's Theorem, Hilbert spaces, orthogonal systems, Fourier series, Bessel's inequality, Parseval's formula, convolutions, Fourier transform, distributions, Sobolev spaces.

References

Apostol, *Mathematical Analysis.*
Knapp, *Basic Real Analysis.*
Riesz-Nagy, *Functional Analysis.*
Royden, *Real Analysis.*
Rudin, *Principles of Mathematical Analysis.*
Rudin, *Real and Complex Analysis.*
Rudin, *Functional Analysis.*
Simmons, *Introduction to Topology and Modern Analysis.*
Wheeden-Zygmund, *Measure and Integration.*
Folland, *Real Analysis.*

Real Analysis I covers the material on calculus, and Lebesgue measure and integration. It is roughly Chapters I-III and V-VI of Knapp. The remaining material is in Real Analysis II.

Complex Analysis I, II – 60370, 60380

I. Winding number, integral along curves. Various definitions of a holomorphic function. Connection with harmonic functions. Cauchy Integral Theorems and Cauchy Integral Formula for closed curves in a domain, and for the boundary of a domain, Poisson Formula. The integral of a holomorphic function and its dependence on the path of integration. Open Mapping Theorem, Inverse Function Theorem, maximum and minimum principle, Liouville's Theorem. Uniform convergence of holomorphic functions. Normal families of holomorphic functions. Montel and Vitali Theorems, Picard's Theorem. Power series, Laurent series. Residues and classification of isolated singularities, meromorphic functions. Divisor of a meromorphic function. Residue Theorem, argument principle, Rouché's Theorem, computation of integrals. Riemann Mapping Theorem, argument principle. Möbius maps. Schwartz Lemma. Theorems of Mittag-Leffler and Weierstrass. Gamma Function, Riemann Zeta Function, Weierstrass $\wp$-function.

II. Definition of complex manifolds and examples**.** Riemann surfaces. The concepts of divisors, line bundles, differential forms and Chern forms. The Riemann-Roch Theorem. The Dirichlet problem for harmonic functions. The concept of genus of a Riemann surface.

References

Ahlfors, *Complex Analysis*.
Burchel, *An Introduction to Classical Complex Analysis I.*
Conway, *Functions of One Complex Variable*.
Forster, *Lectures on Riemann Surfaces.*
Gunning, *Lectures on Riemann Surfaces***.**
Knopp, *Theory of Functions I, II, and Problem Books*.

Complex Analysis I covers approximately Chapters 1-6 of Ahlfors. Complex Analysis II covers the remaining material.

Basic Geometry and Topology (Fall Semester) – 60330

1. Point-set topology (a quick review): topological spaces, subspaces, quotients and products. Properties of topological spaces: Hausdorff, compact, connected. Examples: spheres projective spaces, homogeneous spaces, CW-complexes.

2. Basic algebraic topology: fundamental group, covering spaces, the van Kampen Theorem, the fundamendal group of a surface.

3. Smooth manifolds: tangent bundle and cotangent bundle, vector bundles, constructions with vector bundles (sums, products symmetric and exterior powers), sections of vector bundles, differential forms, the de Rham complex, inverse and implicit function theorems, transversality.

References

Munkres, *Topology*.
Hatcher, *Algebraic Topology (Chapter 1)*.
Lee, *Introduction to Smooth Manifolds*.

Basic Topology II (Spring Semester) – 60440

1. Homology: singular homology, the Eilenberg-Steenrod axioms, homology group of spheres, the degree of a map between spheres, homology calculations via CW complexes, proof of homotopy invariance, proof of excision, universal coefficient and Kunneth Theorem.

2. Cohomology: the cup product, the cohomology ring of projective spaces.

3. Poincare duality.

Reference
Hatcher, *Algebraic Topology*

Basic Differential Geometry (Spring Semester) – 60670

1. Connections in vector bundles: Covariant derivative, parallel transport, orientability, curvature, baby Chern-Weil.

2. Riemannian geometry: Levi Cevita connection, exponential map, Jacobi fields, arc length variation formulas, fundamental equations for metric immersions and submersions, space forms, Hopf-Rinow, Hadamard-Cartan, Bonnet-Myers (Gauss-Bonnet, Bochner technique).

3. Other geometric structures: (one or more of Kahler manifolds, symplectic manifolds, contact manifolds).

References
Chavel, *Riemannian Geometry: A modern introduction.*
Grove, *Riemannian geometry: A metric entrance.*
Petersen, *Riemannian Geometry*.
Gallot, Hulin, and Lafontaine, *Riemannian geometry.*
Cullen, *Introduction to General Topology*.
Dugundji, *Topology*.
Kelley, *General Topology*. Munkres, *Topology*.
Steen, *Counterexamples in Topology*.

Logic I, II – 60510, 60520

1. Model Theory
First order predicate logic. Structures and theories. Compactness theorem. Ultraproducts. Löwenheim-Skolem theorems. Saturation and homogeneity. Quantifier elimination: methods, examples, and applications. Types and type spaces. Prime and Saturated models. Countable models and countable categoricity. Henkin constructions should be seen, for example in the omitting types theorem.
If time permits, more advanced topics may be included.

References
C. C. Chang and H. J. Keisler, *Model Theory*.
D. Marker, *Model theory, an introduction*.
K. Tent and M. Ziegler, *A course in model theory*.

2. Computability Theory
Turing machines. Primitive recursive functions, partial recursive functions, equivalence of Turing and Kleene definitions. Recursive sets, computably enumerable sets. The Recursion Theorem. Index sets and Rice's Theorem. Strong reducibilities (*m*-reducibility, 1-reducibility). Relative computability, Turing degrees, jumps, the Kleene-Post Theorem. The arithmetical hierarchy. Computably enumerable degrees, the Friedberg-Muchnik Theorem, the Low Basis Theorem.
If time permits, further topics may be included.

References
R. I. Soare, *Recursively Enumerable Sets and Degrees.*
H. Rogers, *Theory of Recursive Functions and Effective Computability.*
S. B. Cooper, *Computability Theory*

3. Set Theory
Axioms of *ZFC*, Schröder-Bernstein Theorem, ordinals, ordinal arithmetic, proof and definition by recursion. The Von Neumann hierarchy. Cardinals, cardinal arithmetic. Special kinds of cardinals—regular and singular, successor and limit, inaccessible, equivalent versions of the Axiom of Choice, the constructible hierarchy.
If time permits, further topics may be included.

References
T. Jech, *Set Theory*.
K. Kunen, *Set Theory*.
P. Cohen, *Set Theory and the Continuum Hypothesis.*

Logic I covers the material on model theory and part of the material on set theory, including ordinal and cardinal arithmetic, and definitions by recursion. Logic II covers the material on computability, plus more set theory.

Discrete Mathematics – 60610

This course provides an introduction to the questions of existence, structure and enumeration of discrete mathematical objects. Topics include:

1. Enumeration — basic counting principles (including permutations, combinations, compositions, pigeon-hole principle and inclusion-exclusion), basic counting sequences (such as binomial coefficients, Catalan numbers and Stirling numbers), and recurrence relations and generating functions.

2. Structure and existence — Graphs (including trees, connectivity, Euler trails and Hamilton cycles, matching and coloring), partially ordered sets and lattices, basic Ramsey theory, error detecting and correcting codes, combinatorial designs, and techniques from probability and linear algebra.

Other topics chosen by the instructor may be included if time permits.

The course will be at the level of the following books:

Stasys Jukna, *Extremal combinatorics: With Applications in Computer Science*
Peter Cameron, *Combinatorics: Topics, Techniques, Algorithms*
J.H. van Lint and R.M. Wilson, *A Course in Combinatorics*

Optimization – 60620

Convex sets. Caratheodory and Radon's theorems. Helly's Theorem. Facial stricture of convex sets. Extreme points. Krein-Milman Theorem. Separation Theorem. Optimality conditions for convex programming problems. Introduction to subdifferential calculus. Chebyshev approximations.

References
Barvinok, *A Course in Convexity*.
R. Webster, *Convexity*.

Basic PDE – 60650

Laplace equations: Green's identity, fundamental solutions, maximum principles, Green's functions, Perron's methods. Parabolic equations: Heat equations fundamental solutions, maximum principles, finite difference and convergence, Stefan Problems. First order equations: Characteristic methods, Cauchy problems, vanishing of viscosity-viscosity solutions. Real analytic solutions: Cauchy- Kowalevski theorem, Holmgren theorem.

Reference
L. Evans, *Partial Differential Equations*.